Functional Analysis Review

Filip Belik

December 13, 2022

1 Metric Spaces

A **metric space** is a pair (X, d) of a set X and a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) \geq 0$ with equality if $x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$. This metric can induce a norm $\|\cdot\| = d(\cdot, 0)$.

Theorem 1.1. Hölder Inequality: For $a \in l^p$ and $b \in l^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{\infty} |a_i b_i| \leq \|a\|_{l^p} + \|b\|_{l^q}.$$

Theorem 1.2. Minkowski Inequality: If $a, b \in l^p$, then

$$\|a + b\|_{l^p} \leq \|a\|_{l^p} + \|b\|_{l^p}.$$

A metric space X is said to be **separable** if it contains a countable dense subset. Examples to keep in mind: l^p is separable for $1 \leq p < \infty$ by using finite dimensional representations of sequences with rational entries, but l^∞ is not separable as any number between 0 and 1 can be represented in binary by elements in l^p which are all at least a distance of 1 from each other.

A metric space X is called **complete** if for every sequence (x_n), where each $x_i \in X$, that is Cauchy is also convergent. Note that l^p is complete for all $1 \leq p \leq \infty$. Note that $C[0, 1]$ is incomplete under the metric $d(f, g) = \int_0^1 |f(t) - g(t)| dt$ as one can converge to a discontinuous function with a sequence of continuous functions.

2 Banach Fixed Point Theorem

A **fixed point** of a map $T : X \to X$ is an element $x \in X$ such that $Tx = x$. We call T a **contraction** if there exists a constant $\alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

Theorem 2.1. Contraction Mapping: Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed point $x \in X$ such that for any $x_0 \in X$,

$$\lim_{n\to\infty} T^n x_0 = x.$$

3 Normed and Banach Spaces

A **vector space** over a field K is a set $X \neq \emptyset$ with two operations $+ : X \times X \to X$ and $\times : K \times X \to X$ such that for all $x, y, z \in X$ and $\alpha, \beta \in K$,

a $x + y = y + x$,

b $x + (y + z) = (x + y) + z$,

c there exists $0 \in X$ such that $0 + x = x$,

d there exists $-x \in X$ such that $x + (-x) = 0$,

e $\alpha(\beta x) = (\alpha\beta)x$,

f there exists $1 \in X$ such that $1 \times x = x$,

g $(\alpha + \beta)x = \alpha x + \beta x$,

h and $\alpha(x + y) = \alpha x + \alpha y$.

The **span** of a subset $M \subset X$ is said to be the union of all finite linear combinations of elements in M.

A **Hamel Basis** is a linearly independent subset $B \subset X$ such that span(B)$= X$.

A **norm** is a function $\|\cdot\| : X \to \mathbb{R}^{\geq 0}$ such that for all $x, y \in X$ and $\alpha \in K$,

a $\|x\| = 0$ if and only if $x = 0$,

b $\|\alpha x\| = |\alpha|\|x\|$,

c and $\|x + y\| \leq \|x\| + \|y\|$.

A **Banach Space** is a complete normed vector space.

Theorem 3.1. Let X be a Banach space. A subspace Y of X is complete if and only if Y is closed.

Let (x_n) be a sequence from a normed vector space X, then for the associated series,

$$\sum_{n=1}^{\infty} x_n = x \in X \iff \lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\| = 0$$

in which case we call the series **convergent**. The series is **absolutely convergent** if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Note that if X is not complete, a series can be absolutely convergent but not convergent.

A **Shauder Basis** for a normed vector space X is a sequence (e_n) from X such that for all $x \in X$, there exists a unique sequence of scalars (α_n) such that

$$\sum_{n=1}^{\infty} \alpha_n x_n = x.$$

To keep in mind as a good example, the set of standard basis vectors, $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, in l^p is a Schauder basis for $1 \leq p < \infty$, but not for l^∞ because a finite linear combination of them will always be a distance of 1 away from $(1, 1, 1 \ldots) \in l^\infty$, while $(1, 1, \ldots)$ does not belong to l^p for $p < \infty$.

Theorem 3.2. If a normed vector space X has a Schauder basis, then it is separable.

A metric space X is said to be **compact** (we only use sequential compactness in this course) if every sequence in X has a convergent subsequence in X.

Theorem 3.3. For X a metric space and $M \subset X$, if M is compact then M is closed and bounded. And if X is finite dimensional and M is closed and bounded, then M is compact.

Consider in infinite dimensions the set of basis vectors in l^p. They are indeed bounded with norm 1, and the set is closed as it is discrete, however the set is not compact as no sequence of basis vectors will converge.

Theorem 3.4. Let X be a Banach space. The closed unit ball (containing the interior) is compact if and only if X is finite dimensional.

Theorem 3.5. Riesz: Let X be a Banach Space, and let $Y, Z \subset X$ be subspaces with Y a strict subset of Z and Y closed. Then for all $\theta \in (0,1)$, there exists $z \in Z$ with $\|z\| = 1$ and $\|z - y\| \geq \theta$ for all $y \in Y$.

Theorem 3.6. Continuous image of compact set: Let X, Y be metric spaces and $T : X \to Y$ be a continuous map. Then given $M \subset X$ compact, $T(M)$ is a compact set in Y.

Recall, a map $T : X \to Y$ is **continous** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $\|x - y\|_X < \delta$, we have that $\|Tx - Ty\|_Y < \epsilon$.

A **linear operator** is an operator such that its domain $\mathcal{D}(T)$ and its range $\mathcal{R}(T)$ are vector spaces, and for all $x, y \in \mathcal{D}(T)$ and $\alpha \in K$, $T(\alpha x + y) = \alpha Tx + Ty$.

If a linear operator T is bounded, we can define its **norm** as

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

Theorem 3.7. Boundedness and Continuity: Let X and Y be normed vector spaces and $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator, then

1. T is bounded $\iff$ T is continuous,

2. and T is continuous $\iff$ T is continuous at some $x_0 \in \mathcal{D}(T)$.

Theorem 3.8. Bounded Linear Extensions: Let $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator where Y is Banach and T is bounded. Then there exists an extension $\tilde{T} : \overline{\mathcal{D}(T)} \to Y$ where $\tilde{T}$ is also linear and $\left\|\tilde{T}\right\| = \|T\|$.

A **linear functional** is a linear operator $f : \mathcal{D}(f) \to \mathbb{R}$ (or $\mathbb{C}$).

For a vector space X over $\mathbb{R}$ (or $\mathbb{C}$), denote the **algebraic dual space**, X^*, as the set of all linear functionals with domain X. Note that this forms a vector space. Note that for finite dimensional spaces, they are equivalent (by some mapping that preserves the norm) to their algebraic dual space.

Theorem 3.9. For X a vector space, there exists an injective linear mapping from X to X^{**} where x maps to g_x such that $g_x(f) = f(x)$ for all $f \in X^*$.

We call a space X **reflexive** if $X \backsimeq X^{**}$.

For a normed vector space X over $\mathbb{R}$ (or $\mathbb{C}$), denote the **dual space**, X', as the set of all bounded linear functionals with domain X.

Denote $B(X, Y)$ as the set of bounded linear operators from X to Y. This forms a normed vector space.

Theorem 3.10. If Y is a Banach space, then $B(X, Y)$ is a Banach space. Hence, X' is always a Banach space.

4 Inner Product and Hilbert Spaces

An **inner product space** is a vector space with an **inner product**, $\langle \cdot, \cdot \rangle \to \mathbb{R}$ (or $\mathbb{C}$) such that for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$),

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ and $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$,
3. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0 \iff x = 0$.

In addition, this inner product defines a norm on X given by

$$\|x\|^2 = \langle x, x \rangle$$

and a metric on X given by

$$d(x, y) = \|x - y\|.$$

An inner product defines a notion of angles between vectors where

$$\cos(\theta) = \frac{\langle x, y \rangle}{\sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}}.$$

A **Hilbert Space** is a complete inner product space. l^2 is an example.

Theorem 4.1. Parallelogram Law: Given x, y from an inner product space,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 4.2. Polarization Identity: Given x, y from an inner product space,

$$\|x+y\|^2 - \|x-y\|^2 = 4\,\mathrm{Re}\langle x, y\rangle$$

and

$$\|x+iy\|^2 - \|x-iy\|^2 = 4\,\mathrm{Im}\langle x, y\rangle$$

Theorem 4.3. Cauchy-Schwartz Identity: Given x, y from an inner product space, $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Theorem 4.4. Inner Product is Continuous: If X is an inner product space, and $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

Theorem 4.5. Every Inner Product Space Can Be Completed: For every inner product space X, there exists a Hilbert space H, and a dense subspace $W \subset H$, and an isomorphism $A : X \to W$ such that $\langle Ax, Ay\rangle_H = \langle x, y\rangle_X$ for all $x, y \in X$.

Theorem 4.6. Let Y be a subspace of a Hilbert space H. Then,

1. Y is a Hilbert space $\iff$ Y closed in H,
2. Y finite dimensional $\implies$ Y closed,
3. and H separable $\implies$ Y separable.

Theorem 4.7. Let X be an inner product space and $Y \subset X$ be a closed linear subspace. Then for each $x \in X$, there exists a unique $y \in Y$ such that $\|x - y\|$ is minimized. Moreover, $(x - y) \perp Y$.

Theorem 4.8. Let H be a Hilbert space and $Y \subset H$ be a closed linear subspace. Then every $h \in H$ has a unique decomposition as $h = x + y$ where $y \in Y$, $x \in Y^\perp$. We denote this as $H = Y \bigoplus Y^\perp$.

Let X be an inner product space and $M \subset X$. We define the perpendicular set to M as

$$M^\perp := \{x \in X : \langle x, m\rangle = 0 \text{ for all } m \in M\}.$$

Theorem 4.9. Let X be an inner product space and $M \subset X$. Then

1. $M^{\perp}$ is always a vector space,
2. $M \subset M^{\perp\perp}$,
3. and if M is a closed subspace of a Hilbert space, so Hilbert itself, then $M^{\perp\perp} = M$.

Given a Hilbert space H, a set $A \subset H$ is said to be **orthogonal** if $\langle x, y \rangle = 0$ for all $x, y \in A$ where $x \neq y$. And it is said to be orthonormal if it is orthogonal, and $\langle x, x \rangle = \|x\|^2 = 1$ for all $x \in H$.

Theorem 4.10. Bessel's Inequality: Let H be a Hilbert space and (e_n) be an orthonormal sequence in H. Then for every $x \in X$,

$$\sum_{n=1}^{\infty} \langle x, e_n \rangle^2 \leq \|x\|^2.$$

Theorem 4.11. Let (e_n) be an orthonormal sequence in a Hilbert space H, then

1. $\sum_{n=1}^{\infty} \alpha_n e_n$ converges $\iff$ $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges, i.e. $(\alpha_n) \in l^2$,
2. if $\sum_{n=1}^{\infty} \alpha_n e_n = x$, then $\alpha_n = \langle x, e_n \rangle$,
3. and for any $x \in H$, $\sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ converges.

A **total set** in a normed space X is a subset $M \subset X$ such that $\operatorname{span}(M)$ is dense in X.

Theorem 4.12. Totality: Let M be a subset of an inner product space X.

1. If M is total, then $x \perp M \implies x = 0$.
2. If X is a Hilbert space, then M total $\iff$ $M^{\perp} = \{0\}$.

Theorem 4.13. Parseval's Identity: An orthonormal set M of a Hilbert space H is total if and only if $\|x\|^2 = \sum_{e_k \in M_x} |\langle x, e_k \rangle|^2$ for all $x \in H$ where $e_k \in M_x$ denotes the elements in M such that $\langle x, e_k \rangle \neq 0$. Hence, the set M_x is at most countable for every x.

Theorem 4.14. Separable Hilbert Spaces: Let H be a Hilbert space. Then

1. H is separable $\implies$ every orthonormal set in H is countable.
2. If there exists a countable total orthonormal set $M \subset H$, then H is separable.

Theorem 4.15. Riesz's Theorem: Let H be a Hilbert space and $f \in H'$. Then there exists a unique $z \in H$ such that $f(x) = \langle x, z \rangle$ and $\|f\| = \|z\|$. In words, all bounded linear functionals on a Hilbert space are inner products.

Theorem 4.16. In a Hilbert space, if $\langle x, z_1 \rangle = \langle x, z_2 \rangle$ for all $x \in H$, then $z_1 = z_2$.

Theorem 4.17. Riesz-Representation Theorem: Let H_1, H_2 be Hilbert spaces and $h : H_1 \times H_2 \to \mathbb{R}$ (or $\mathbb{C}$) be a bounded sesquilinear form (linear in first argument, conjugate linear in second). Then there exists a unique $S \in B(H_1, H_2)$ such that $h(x, y) = \langle Sx, Sy \rangle$. Moreover, $\|h\| = \|S\|$. In words, every bounded linear operator is a bounded sesquilinear form.

Given Hilbert spaces H_1, H_2 and $T \in B(H_1, H_2)$, the **Hilbert adjoint operator** is the operator $T^* \in B(H_2, H_1)$ is such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in H_1$, $y \in H_2$.

Theorem 4.18. The Hilbert adjoint operator T^* of T exists, is unique, and is bounded with norm $\|T^*\| = \|T\|$. In addition,

1. $T^{**} = T$,
2. $(\alpha T)^* = \overline{\alpha} T^*$,
3. $(S + T)^* = S^* + T^*$,
4. $\|T^* T\| = \|T T^*\| = \|T\|^2$,
5. and if ST makes sense, then $(ST)^* = T^* S^*$.

Let $T : H \to H$ be a bounded linear operator. We say T is **self-adjoint** of **hermitian** if $T = T^*$, **unitary** if $T^* = T^{-1}$, and **normal** if $TT^* = T^* T$.

5 Fundamental Theorems for Normed and Banach Spaces

Theorem 5.1. Hahn-Banach on a Vector Space: Assume X is a vector space, $Z \subset X$ is a linear subspace, $\rho : X \to \mathbb{R}$ is a sublinear functional ($\rho(\alpha x) = |\alpha|\rho(x)$, $\rho(x+y) \le \rho(x) + \rho(y)$), and $f : Z \to \mathbb{R}$ such that $|f(z)| \le \rho(z)$ for all $z \in Z$. Then there exists $\tilde{f} : X \to \mathbb{R}$ such that $|\tilde{f}(x)| \le \rho(x)$ for all $x \in X$ and $f(z) = \tilde{f}(z)$ for all $z \in Z$.

Theorem 5.2. Hahn-Banach on a Normed Vector Space: Assume X is a normed vector space, $Z \subset X$ is a linear subspace, and let $f \in Z'$. Then there exists $\tilde{f} \in X'$ such that $\tilde{f}(z) = f(z)$ for all $z \in Z$ and $\|f\|_Z = \left\|\tilde{f}\right\|_X$.

Theorem 5.3. Uniform Boundedness: Let X and Y be normed vector spaces, and X is Banach. If (T_n) is a sequence of operators in $B(X,Y)$, and for all $x \in X$ there exists $c_x \in \mathbb{R}$ such that $\sup_n \|T_n x\| < c_x$, then there exists $c > 0$ such that $\sup_n \|T_n\| < c$.

We call a vector space X **reflexive** if $X'' \sim X$, i.e. there exists a bijection between the two that preserves norm. Consider $C : X \to X''$ by $(Cx)(f) = f(x)$, we can show C preserves norm and is one-to-one, so X is reflexive $\iff$ C is onto.

Theorem 5.4. Let X be a normed vector space, then X is reflexive implies X is Banach.

Theorem 5.5. X' is separable implies that X is separable.

Let X be a normed vector space, and (x_n) be a sequence of elements of X. Then for $x \in X$,

1. $x_n \to x$ strongly $\iff$ $\|x - x_n\| \to 0$,
2. and $x_n \to_w x$ (weakly) $\iff$ $f(x_n) \to f(x)$ for all $f \in X'$.

Let X and Y be metric spaces. A mapping $T : \mathcal{D}(T) \subset X \to Y$ is an **open mapping** if any open set in $\mathcal{D}(T)$ is mapped into an open set in Y.

Theorem 5.6. Open Mapping Theorem: Let X, Y be Banach spaces and $T \in B(X,Y)$ such that T is onto. Then T is an open mapping, and if T is one-to-one (hence bijective), then T^{-1} is continuous and hence bounded.

Let X and Y be normed metric spaces and $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator. Define the **graph** of T as

$$\mathcal{G}(T) := \{(x, Tx) \in X \times Y : x \in \mathcal{D}(T)\}.$$

We say that T is a **closed operator** if $\mathcal{G}(T)$ is a closed set in $X \times Y$.

Theorem 5.7. Closed Graph Theorem: Let X and Y be Banach spaces and $T : \mathcal{D}(T) \subset X \to Y$ be a closed linear operator. Then if $\mathcal{D}(T)$ is closed in X, then T is bounded.

Theorem 5.8. Closed Linear Operator: Let $T : \mathcal{D}(T) \to Y$ be a linear operator where X and Y are normed vector spaces. Then T is closed if and only if it satsifies the following property. If $x_n \to x$ where $x_n \in \mathcal{D}(T)$, and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

6 Spectral Theory of Linear Operators in Normed Spaces

Use Figure 1 to identify the spectrum $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$ and the residual $\rho(T)$. The important thing to note in infinite dimensions is that for maps from spaces to themselves, injectivity does not imply surjectivity.

Theorem 6.1. Let X be a Banach space, $T \in B(X, X)$. If $\|T\| < 1$, then $(I - T)^{-1}$ exists as a bounded linear operator defined on all of X, and

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n = I + T + T^2 + \cdots .$$

Theorem 6.2. If X Banach, and $T \in B(X, X)$, then $\rho(T)$ is open in $\mathbb{C}$, and if $\lambda_0 \in \rho(T)$, then the resolvent is given by

$$(T - \lambda I)^{-1} =: R_\lambda(T) = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^i R^i_{\lambda_0}.$$

Theorem 6.3. Bounded Spectrum on Banach Space is Compact: Let X be Banach and $T \in B(X, X)$. Then for all $\lambda \in \sigma(T)$, then $|\lambda| \le \|T\|$, so $\sigma(T)$ is a compact subset of $\mathbb{C}$.

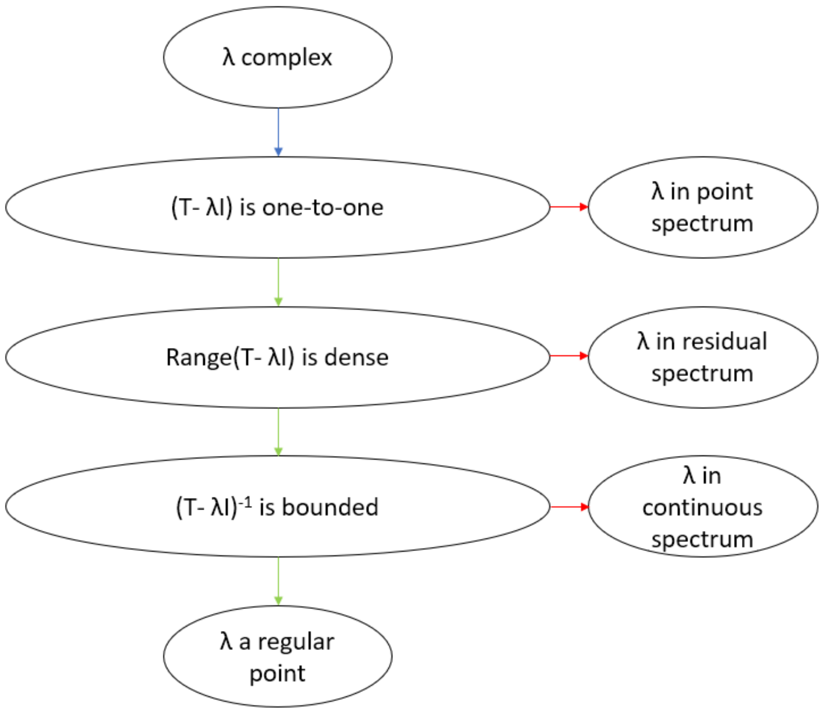

Figure 1: Flowchart for identification of spectrum. Green arrows denote answers of yet, red arrows denote answers of no.

Define the **spectral radius** of an operator $T \in B(X,X)$ for X Banach by

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Theorem 6.4. Let X be Banach and $T \in B(X,X)$, and $\mu, \lambda \in \sigma(T)$. Then

1. $R_\mu - R_\lambda = (\mu - \lambda)(R_\mu R_\lambda)$ (composition),
2. If $ST = TS$, then $R_\lambda S = S R_\lambda$,
3. and $R_\lambda R_\mu = R_\mu R_\lambda$ (composition).

7 Compact Linear Operators on Normed Spaces and Their Spectrum

Let X, Y be normed vector spaces and $T : X \to Y$ be linear. T is a **compact linear operator** if for all bounded $M \subset X$, $\overline{T(M)}$ is compact in X.

Theorem 7.1. A compact linear operator is bounded.

Theorem 7.2. Alternative Characterization of Compactness: Let X, Y be normed vector spaces and $T : X \to Y$ be linear. Then T is compact if and only if for every bounded (x_n), the sequence (Tx_n) has a convergent subsequence.

Theorem 7.3. Compact linear operators form a normed vector space by closure under addition and scalar multiplication.

Theorem 7.4. Compact Identification: Let X, Y be normed vector spaces and $T : X \to Y$ be linear. Then

1. T bounded and $\dim(T(X)) < \infty \implies T$ compact.
2. $\dim(X)$ finite $\implies T$ compact.

Theorem 7.5. Finite Compact Approximation: Let X, Y be normed vector spaces, Y Banach, and $T : X \to Y$ be linear. And let (T_n) be a sequence of compact operators from X to Y. If $\|T_n - T\| \to 0$ then T is compact.

Theorem 7.6. Weak Convergence with Compact Operators: Let X, Y be normed vector spaces and $T : X \to Y$ be a compact and linear operator. Given (x_n) such that $x_n \to^w x$, then Tx_n converges strongly in Y.

Theorem 7.7. Let X be a normed vector space, $T : X \to X$ be compact and linear. Then for all $\epsilon > 0$, the set

$$\{\lambda \in \sigma_p(T) : |\lambda| > \epsilon\}$$

is finite. So 0 is the only possible accumulation point of $\sigma_p(T)$. In addition, if $\lambda \in \sigma_p(T)$ is nonzero, then $\mathcal{N}(T - \lambda I)$ is finite dimensional, so only finite dimensional eigenspace. And lastly, if $0 \in \sigma(T)$, then $0 \in \sigma_p(T)$.

Theorem 7.8. Alternative Statement of Fredholm Alternative: Let X be a normed vector space and $T : X \to X$ be compact and linear and $\lambda \in \mathbb{C}$ nonzero. Then $(T - \lambda I)$ is one-to-one if and only if $(T - \lambda I)$ is onto.

8 Spectral Theory of Bounded Self-Adjoint Linear Operators

Let X, Y be normed vector spaces and $T : X \to Y$ be bounded and self-adjoint.

Define the **Rayleigh quotient**

$$q(x) = \frac{\langle Tx, x\rangle}{\langle x, x\rangle}$$

Then we know the following things about T and its spectrum:

1. $\sigma(T) \subset \mathbb{R}$ and $\sigma_r(T) = \emptyset$,
2. $\lambda_1, \lambda_2 \in \sigma_p(T)$, and $\lambda_1 \neq \lambda_2$ implies $\mathcal{N}(T - \lambda_1 I) \perp \mathcal{N}(T - \lambda_2 I)$,
3. $\sigma(T) \subset [\inf_{x\neq 0} q(x), \sup_{x\neq 0} q(x)]$,
4. $\inf_{x\neq 0} q(x), \sup_{x\neq 0} q(x) \subset \sigma(T)$,
5. $\|T\| = \sup_{x\neq 0} |q(x)|$.

Theorem 8.1. Alternative Characterization of Resolvent: Let H be a Hilbert space and $T : H \to H$ be bounded and self-adjoint. Then $\lambda \in \rho(T) \implies$ there exists $c > 0$ such that $\|(T - \lambda I)x\| \geq c\|x\|$.

Properties of $T : H \to H$ compact:

1. For all $\epsilon > 0$, the set $\{\lambda \in \sigma_p(T) : |\lambda| > \epsilon\}$ is finite,
2. $\lambda \in \sigma_p(T)$ nonzero implies that $\mathcal{N}(T - \lambda I)$ is finite dimensional,
3. and $\lambda \in \sigma(T)$ nonzero implies $\lambda \in \sigma_p(T)$.

Theorem 8.2. Spectral Theorem: Let H be a separable Hilbert space, $T : H \to H$ be compact and self-adjoint. Then H has a complete orthonormal basis consisting of eigenvectors of T.