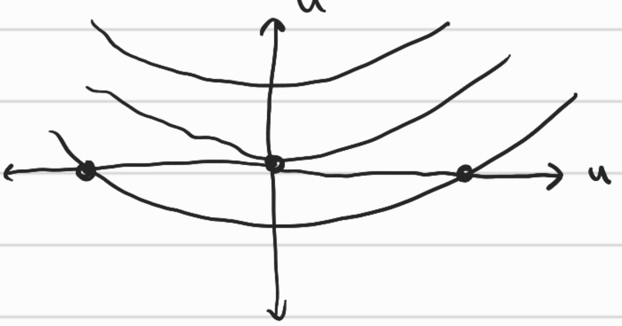
1/9/23:

A **bifurcation** is a transition from one state to another. **Bifurcation theory** is the study of equations of form

$$f(u, \lambda) = 0.$$

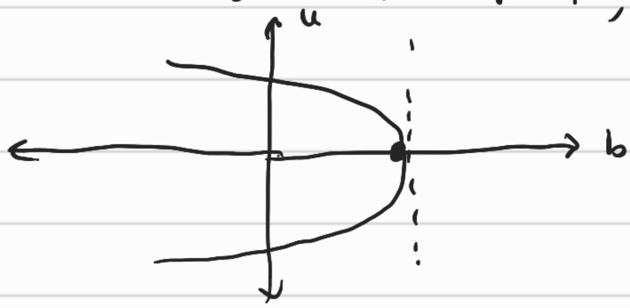
Continuation methods: Consider $\frac{du}{dt} = u^2 + au + b$



Could have 0, 1, 2 solutions. Depends on discriminant $a^2 - 4b$.

$a^2 - 4b < 0 \Rightarrow$ no real roots, $a^2 - 4b > 0 \Rightarrow 2$ real roots

Or we can make bifurcation plots, fix a,



We will use XPP for simulating dynamical systems.

1/11/23:

Suppose you have polynomials $f(x)$, $g(x)$ of orders n and m respectively, then we can write

$$f(x) = \alpha \prod_{i=1}^{n} (x - \lambda_i), \quad g(x) = \beta \prod_{j=1}^{m} (x - \mu_j)$$

where λ_i's are roots of f, μ_j's are roots of g. Then the **resultant** of f & g is given by

$$R(f, g) = \alpha^m \beta^n \prod_{1 \le i \le n,\, 1 \le j \le m} (\lambda_i - \mu_j).$$

Note that if f, g share a root, then $R(f,g) = 0$.
To test if f has a repeated root, we check if $R(f, f') = 0$. Can be done in Maple.

<u>1/13/23:</u>

<u>Continuation Theory:</u> Consider a problem $F(u, \lambda) = 0$ where $u \in H$, a Hilbert space, $\lambda \in \mathbb{R}$, $F: H \times \mathbb{R} \to H$ ($\in C^2$ or C^∞).

<u>Ex:</u> $F(u, \lambda) = u'' + \lambda u e^{u} = 0$ given $u(0) = u(1) = 0$.

Suppose we know a solution $F(u_0, 0) = 0$. Can we extend this solution?

- First guess: Try perturbation, a power series.

$$F(u(\varepsilon), \varepsilon) = 0.$$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

Then
$$\begin{aligned}
0 &= F(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \varepsilon) \\
&= F(u_0 + \varepsilon \Delta u, \varepsilon) \\
&= F(u_0, 0) + F_u(u_0, 0)\,\varepsilon \Delta u + F_\varepsilon(u_0, 0)\,\varepsilon \\
&\quad + F_{uu}(u_0, 0)\frac{(\varepsilon \Delta u)^2}{2} + F_{u\varepsilon}(u_0, 0)(\varepsilon \Delta u)^2 + F_{\varepsilon\varepsilon}(u_0, 0)\,\varepsilon^2 \\
&\quad + \cdots \\
&= \overset{\nearrow 0}{F(u_0, 0)} + F_u(u_0, 0)\,\varepsilon u_1 + \varepsilon F_\varepsilon(u_0, 0) + O(\varepsilon^2)
\end{aligned}$$

So require $F_u(u_0, 0)\, u_1 + F_\varepsilon(u_0, 0) = 0$

$$\Rightarrow u_1 = -F_u^{-1}(u_0, 0)\, F_\varepsilon(u_0, 0)$$

<u>1/18/23</u>

<u>Implicit Function Theorem:</u> Given F, u_0 s.t. $F(u_0, 0) = 0$ and $F_u^{-1}(u_0, 0)$ exists, then there exists a smooth function $u = u(\varepsilon)$ with $u(0) = u_0$ s.t. $F(u(\varepsilon), \varepsilon) = 0$.

<u>Proof:</u> Let $u = u_0 + \varepsilon V$, and then we have that

$$F(u, \varepsilon) = F(u_0 + \varepsilon V, \varepsilon) = F(u_0, 0) + F_u(u_0, 0)\,\varepsilon V + F_\varepsilon(u_0, 0)\,\varepsilon + G(\varepsilon V, \varepsilon)$$

where $G(\varepsilon V, \varepsilon)$ is a remainder term of order ε^2 by definition.

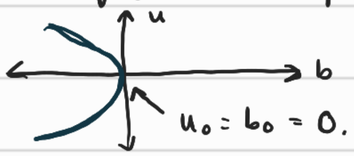
We know $F(u_0, 0) = 0$, and we want $F(u, \varepsilon) = 0$, so rewrite

$$F_u(u_0, 0)\,\varepsilon V = -F_\varepsilon(u_0, 0)\,\varepsilon - G(\varepsilon V, \varepsilon)$$

$$\Rightarrow \varepsilon v = -F_u^{-1}(u_0,0)\left[-F_\varepsilon(u_0,0)\,\varepsilon - G(\varepsilon v,\varepsilon)\right]$$
$$\Rightarrow v = -F_u^{-1}(u_0,0)\left[-F_\varepsilon(u_0,0) - \tfrac{1}{\varepsilon}G(\varepsilon v,\varepsilon)\right]$$
$$= M(v).$$

We prove $v = M(v)$ has a solution through the contraction mapping thm.
Well, $\|M(v)-M(w)\| = \|\frac{1}{\varepsilon}G(\varepsilon v,\varepsilon) - \frac{1}{\varepsilon}G(\varepsilon w,\varepsilon)\|$, we then use the identity: $H(v)-H(w) = (v-w)\int_0^1 H_u(sv-(1-s)w)\,\partial s$.
So $\|M(v)-M(w)\| = \|v-w\|\,\|\int_0^1 \frac{1}{\varepsilon}G_u(\varepsilon sv-(1-s)\varepsilon w,\varepsilon)\,\partial s\|$
$\leq \varepsilon K\|v-w\|$ since G is order ε^2, so making ε sufficiently small, we have $\|M(v)-M(w)\| \leq \delta\|v-w\|$ for $0<\delta<1$.

What if $F_u^{-1}(u_0,\varepsilon)$ does not exist? Example: Limit points.



$u^2 + b = 0 \Rightarrow u = \pm\sqrt{b}$,
Doing power series, $f_u = 2u$,
$f_u(u_0,0) = 2(0) = 0$, <u>not invertible</u>

Introduce a new parameter s, wish to solve
$u = u_0 + u_1 s + u_2 s^2 + \cdots$, $\quad \varepsilon = 0 + \varepsilon_1 s + \varepsilon_2 s^2 + \cdots$.

$$F(u,\varepsilon) = F(u_0 + u_1 s + u_2 s^2 + \cdots,\ 0 + \varepsilon_1 s + \varepsilon_2 s^2 + \cdots)$$
$$= F(u_0,0) + F_u(u_0,0)(u_1 s + u_2 s^2 + \cdots)$$
$$+ F_\varepsilon(u_0,0)(\varepsilon_1 s + \varepsilon_2 s^2 + \cdots)$$
$$+ F_{uu}(u_0,0)(u_1 s + u_2 s^2 + \cdots)^2/2$$
$$+ F_{u\varepsilon}(u_0,0)(u_1 s + u_2 s^2 + \cdots)(\varepsilon_1 s + \varepsilon_2 s^2 + \cdots)$$
$$+ F_{\varepsilon\varepsilon}(u_0,0)(\varepsilon_1 s + \varepsilon_2 s^2 + \cdots)^2/2$$
$$+ O(s^3).$$

$O(1) = F(u_0,0) = 0$ ✓
$O(s) = F_u(u_0,0)u_1 + F_\varepsilon(u_0,0)\varepsilon_1$
$O(s^2) = F_u(u_0,0)u_2 + F_\varepsilon(u_0,0)\varepsilon_2 + F_{uu}(u_0,0)\frac{u_1^2}{2}$
$\quad + F_{u\varepsilon}(u_0,0)u_1\varepsilon_1 + F_{\varepsilon\varepsilon}(u_0,0)\frac{\varepsilon_1^2}{2} = 0$

$\vdots \qquad \vdots$

If $u_1 = \Phi$ is in the nullspace of $F_u(u_0,\varepsilon)$, then $\varepsilon_1 = 0$, and then $O(s^2) = F_u(u_0,0)u_2 + F_\varepsilon(u_0,0)\varepsilon_2 + F_{uu}(u_0,0)\frac{u_1^2}{2} = 0$

$$\Rightarrow F_u(u_0,0)\,u_2 = -\left[F_\varepsilon(u_0,0)\,\varepsilon_2 + F_{uu}(u_0,0)\,\frac{u_1^2}{2}\right],$$

Apply the <u>Fredholm Alternative Theorem</u>: $Lu=f$ has a solution iff $\langle f,\psi\rangle = 0 \;\; \forall \;\; \psi \in N(L^*)$. So our condition of solvability is

$$\left\langle F_\varepsilon(u_0,0)\,\varepsilon_2 + F_{uu}(u_0,0)\frac{\phi^2}{2},\ \psi\right\rangle = 0$$

$$\Rightarrow \varepsilon_2 = \frac{-\left\langle F_{uu}(u_0,0)\frac{\phi^2}{2},\ \psi\right\rangle}{\langle F_\varepsilon(u_0,0),\ \psi\rangle}.$$

Then we have $u = s\phi + O(s^2)$, $\varepsilon = s^2\varepsilon_2 + \cdots$.

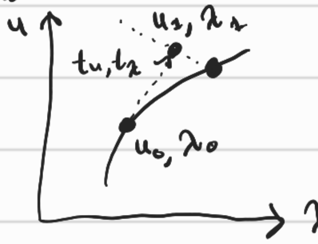
We've parameterized our continuation.

<u>1/20/23:</u>

Again, suppose $F(u_0,\lambda_0)=0$, and we wish to find u_1 s.t. $F(u_1,\lambda_1)=0$ where λ_1 is near λ_0. If $F_u(u(\lambda),\lambda)=0$, then differentiating, $F_u(u_0,\lambda_0)\,u' + F_\lambda(u_0,\lambda_0) = 0 \Rightarrow u' = -F_u^{-1}(F_\lambda(u_0,\lambda_0))$. So we can guess $u_1 = u_0 + u'(\lambda_1-\lambda_0)$.

Iterative method: $x_{j+1} = x_j + \delta_x$ where $F_u(x_j,\lambda)\,\delta x = -F_\lambda(x_j,\lambda)$.

Or, create new variable s, and we wish to find $u'(s)$, $\lambda'(s)$ s.t. $F_u(u(s_0),\lambda(s_0))\,t_u = -F_\lambda(u(s_0),\lambda(s_0))\,t_\lambda$ where $\begin{pmatrix} t_u \\ t_\lambda \end{pmatrix}$ is the tangent vector in u-λ space. We then wish to travel perpendicular to $\begin{pmatrix} t_u \\ t_\lambda \end{pmatrix}$ to find a point (u,λ) s.t. $F(u,\lambda)=0$.



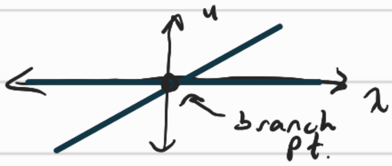
$$G(u,\lambda) = \begin{pmatrix} F(u,\lambda) \\ t_u\cdot(u-u_1) + t_\lambda\cdot(\lambda-\lambda_1) \end{pmatrix}.$$

We wish to find u,λ s.t. $G(u,\lambda) = \begin{pmatrix}0\\0\end{pmatrix}$. So calculate t_u, t_λ, normalize, and use Newton's method on G to find continuation (can show Jacobian of G invertible).

This method still succeeds even if F_u^{-1} DNE.

Both MATCONT (MATLAB) and XPP use this method, called <u>arclength continuation</u>.

<u>Ex:</u> $f(u,\lambda) = u^2 - \lambda u$. We see for all λ, $u=0$ (or λ). And $f_u = 2u-\lambda$ and $f_\lambda = -u$. f_u not invertible at $(0,0)$. $(0,0)$ a branch pt.



Similar Ex: Bratu's Eqn.

$F(u,\lambda) = u'' + \lambda u e^{u} = 0, \quad u(0) = u(1) = 0$

results in Branch points. $u_0 \equiv 0$, $\lambda = 0$.

We expand about u_0: $F(u_0+\varepsilon v, \lambda) = \overset{=0}{F(u_0,\lambda)} + F_u(u_0,\lambda)\,\varepsilon v$

$$\left.\begin{aligned} &= \varepsilon v\left[u_0'' + \lambda\left(e^{u_0} + u_0 e^{u_0}\right)\right] \\ &= \varepsilon v'' + \varepsilon\lambda\left(e^{u_0} + u_0 e^{u_0}\right)v \end{aligned}\right\} \text{Verify by Power series}$$

Given $u_0 \equiv 0$, $F(u_0 + \varepsilon v, \lambda) = \varepsilon(v'' + \lambda v)$, given bdry conditions, we see that $\lambda = n^2\pi^2$, $v(t) = \sin(n\pi t)$.

1/23/23:

Revisit Bratu: $u'' + \lambda u e^{u} = 0$, $u(0) = u(1) = 0$.

Define the Frechet Derivative $\displaystyle \lim_{\varepsilon \to 0} \frac{f(u+\varepsilon v) - f(u)}{\varepsilon}$.

For our ex: $\displaystyle \lim_{\varepsilon\to 0} \frac{u'' + \varepsilon v'' + \lambda(u+\varepsilon v)e^{u+\varepsilon v} - u'' - \lambda u e^{u}}{\varepsilon}$

$$= \lim_{\varepsilon\to 0} \frac{\varepsilon v'' + \lambda\left[u\left(e^{u+\varepsilon v} - e^{u}\right) + \varepsilon v e^{u+\varepsilon v}\right]}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\varepsilon v'' + \lambda\left[u\left(e^{u}\left(1+\varepsilon v + \frac{\varepsilon^2 v^2}{2} + \cdots\right) - e^{u}\right) + \varepsilon v e^{u}\left(1 + \varepsilon v + \frac{\varepsilon^2 v^2}{2} + \cdots\right)\right]}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} v'' + \lambda\left[u e^{u}\left(v + \frac{\varepsilon v^2}{2} + \cdots\right) + v e^{u}\left(1 + \varepsilon v + \frac{\varepsilon^2 v^2}{2} + \cdots\right)\right]$$

$= v'' + \lambda e^{u}(u+1)v$ (directional derivative in "direction" v)

which is a linear operator acting on another function v.

Now, for $u'' + \lambda u e^{u} = 0$, $u(0) = u(1) = 0$, wish to find $u(\lambda)$.

Recall $u \equiv 0$ a soln for all λ.

Implicit Function Theorem: $u = 0$, $\lambda_0 \in \mathbb{R}$, look if linearization, in terms of u, is invertible. $L(u)v = v'' + \lambda e^{u}(u+1)v$, so $L(0)v = v'' + \lambda v$.

Is the nullspace trivial? $v'' + \lambda v = 0$ subject to $v(0) = v(1) = 0$.

Nontrivial if $\lambda = n^2\pi^2$ and $v(x) = \sin(\sqrt{\lambda}\,x) = \sin(n\pi x)$. We know Branch points occur at $\lambda = n^2\pi^2$ for $n \in \mathbb{N}$, can we show it?

Power series: $u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$, $\lambda = \pi^2 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots$

Try to find non-trivial soln in Maple.

$u_1 = \sin(\pi x)$, $\lambda_1 = -8\pi/3$, u_2 is a more complicated fn, λ_2 as well.

Key: Use FAT to determine λ_i's, dsolve to determine $u_i(x)$'s.

1/25/23: $\lambda = \pi^2 - \frac{8\pi}{3}\varepsilon + O(\varepsilon^2)$

$u(x) = \varepsilon \sin(\pi x) + O(\varepsilon^2)$.

Since λ, u are linear in ε, this is a transverse bifurcation, We can find new bifurcations at $\lambda = n^2\pi^2$ for $n \in \mathbb{N}$.

Taking $\lambda_0 = 4\pi^2$, we get solutions

$$\lambda = 4\pi^2 + \frac{11\pi^2}{6}\varepsilon^2 + O(\varepsilon^3).$$

$$u(x) = \sin(2\pi x)\varepsilon + \left(\frac{-1}{2} + \frac{2\cos(2\pi x)}{3} - \frac{\cos(4\pi x)}{6}\right)\varepsilon^2 + O(\varepsilon^3)$$

Parabolic nature of λ tells us we have a pitchfork bifurcation in direction of positive λ. Once more, take $\lambda_0 = 9\pi^2$.

$$\lambda = 9\pi^2 - 8\pi\varepsilon + O(\varepsilon^2)$$
$$u(x) = \sin(3\pi x)\varepsilon + O(\varepsilon^2)$$

Transversal bifurcation again.

Claim: Alternates transversal/pitchfork by even/odd.

Branching Theory: Wish to solve $F(u,\lambda) = 0$, and we know $f(0,\lambda) = 0 \ \forall \lambda$. We look for points where $F_u(0,\lambda)$ not invertible, i.e. $F_u(0,\lambda)\phi = 0$ for $\phi \neq 0$. We will assume $\dim N(F_u(0,\lambda_0)) = 1$.

→ Background: FAT. Let $L: H \to H$ be a bounded operator on a Hilbert space w/ closed range. $Lu = f$ has (at least 1) solution iff $\langle v, f\rangle = 0 \ \forall \ v \in N(L^*)$ (nullspace of adjoint operator, $L^* v = 0$)

Pf) $\Rightarrow$) $Lu = f$ & $\psi \in N(L^*) \Rightarrow 0 = \langle L^*\psi, u\rangle = \langle \psi, Lu\rangle = \langle \psi, f\rangle$

$\Leftarrow$) Suppose $\langle \psi, f\rangle = 0 \ \forall \ \psi \in N(L^*)$. Write $f = f_r + f_\perp$ where $f_r \in R(L)$, $f_\perp \perp R(L)$. Then $\langle f_\perp, Lu\rangle = 0 \ \forall u \Rightarrow \langle L^* f_\perp, u\rangle = 0 \ \forall u$ $\Rightarrow L^* f_\perp = 0 \Rightarrow f_\perp \in N(L^*)$. So by assumption, $\langle f_\perp, f\rangle = 0 \Rightarrow f_\perp = 0$ $\Rightarrow f \in R(L) \Rightarrow \exists$ a solution.

So $f(u,\lambda) = \cancel{f(0,\lambda_0)} + f_u(0,\lambda_0)u + \cancel{f_\lambda(0,\lambda_0)(\lambda-\lambda_0)} + f_{u\lambda}(0,\lambda_0)u(\lambda-\lambda_0) + R(u,\lambda)$

where $R(u,\lambda) = O(u^2, u(\lambda-\lambda_0)^2)$. Denote $f_u^0 = f_u(0,\lambda_0)$.

Let $\phi \in N(f_u^0)$, $\psi \in N(f_u^{0*})$ s.t. $\langle \phi, \psi \rangle = 1$. Now define the projection operator $P(u) = \langle \psi, u \rangle \phi$. Can prove $P^2 = P$.

<u>1/27/23:</u> Expanding $F(u,\lambda) = F_u^0 u + F_{u\lambda}^0 u\lambda + R(u,\lambda)$. Now, We set $P(F)$, $(1-P)(F)$ equal to zero, and let $u = \varepsilon(\phi + w)$ where $w \perp \psi$. We then get a system of equations for w, λ. Let $\mu = (\lambda - \lambda_0)$.

$$G\begin{pmatrix} w \\ \lambda \end{pmatrix} = \begin{pmatrix} \langle \psi, F_{u\lambda}(\phi + w)\mu \rangle + \langle \psi, \frac{R}{\varepsilon} \rangle \\ F_u^0 w + F_{u\lambda}^0(\phi + w)\mu + \frac{R}{\varepsilon} - \langle \psi, F_{u\lambda}^0(\phi + w)\mu + \frac{R}{\varepsilon} \rangle \phi \end{pmatrix} = \vec{0}$$

Note that $\frac{R}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Assuming $\varepsilon, \mu, |w| \ll 1$, $G = \vec{0}$ satisfied.
We then linearize G, set $u, \mu = 0$, and obtain the Jacobian

$$J(G) = \begin{pmatrix} 0 & \langle \psi, F_{u\lambda}^0 \phi \rangle \\ F_u^0 & F_u^0 \phi - \langle \psi, F_u^0 \phi \rangle \phi \end{pmatrix} = \begin{pmatrix} \frac{\partial G_1}{\partial w} & \frac{\partial G_1}{\partial \lambda} \\ \frac{\partial G_2}{\partial w} & \frac{\partial G_2}{\partial \lambda} \end{pmatrix}$$

So, given $\langle \psi, F_{u\lambda}^0 \phi \rangle \neq 0$, $J(G)$ is invertible.

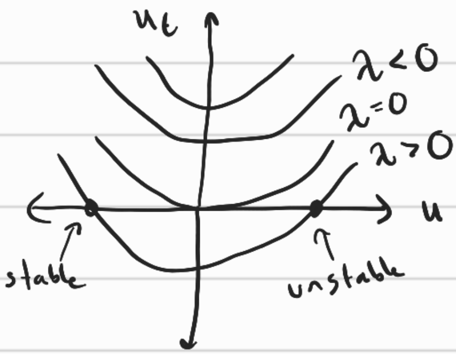
<u>Theorem:</u> If F_u^0 has a one-dimensional nullspace, $F_u^0 \phi = 0$ and $F_u^{0*} \psi = 0$ with $\langle \phi, \psi \rangle = 1$, and $\langle \psi, F_{u\lambda}^0 \phi \rangle \neq 0$, then there exists a nontrivial branch of solutions.

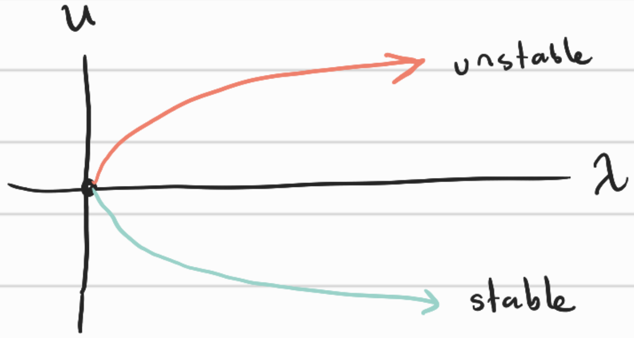
Also, if $\dim(N(F_u^0))$ is odd, $\exists$ at least 1 nontrivial branch of solns.

<u>1/30/23:</u>

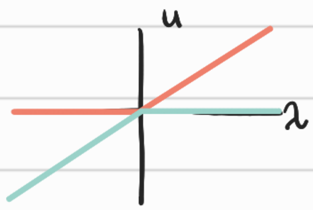
Today, consider problems of form $u_t = F(u,\lambda)$. Steady states occur at $0 = u_t = F(u,\lambda)$. We'll study stability of steady sts.

<u>ex:</u> $u_t = u^2 - \lambda$.

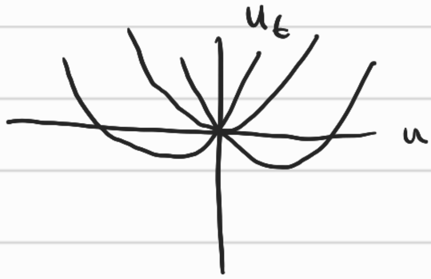




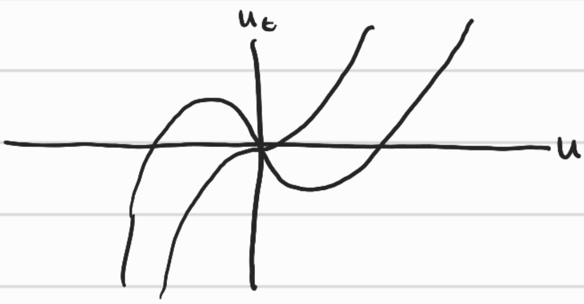
<u>ex:</u> $u_t = u(u - \lambda)$. SS: $u = 0$, $u = \lambda$.

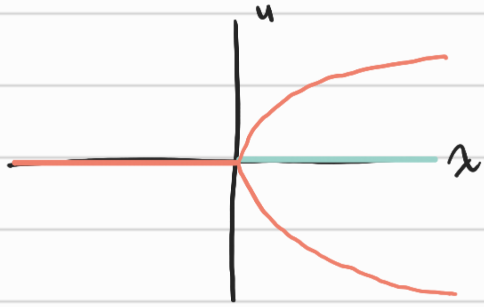


We see an exchange of stability at $\lambda = 0$.



Ex: $u_t = u(u^2 - \lambda)$. $u = 0$, $u^2 = \lambda$





So back to $u_t = F(u, \lambda)$. Linearizing about a known solution, we get $\mu U = F_u(u, \lambda) U$. Now, parameterize the problem by ε.

Case 1: Limit Pts: $\exists\, u_0, \lambda_0$ s.t. $F(u_0, \lambda_0) = 0$, $F_u(u_0, \lambda_0)\phi = 0$, $F_\lambda(u_0, \lambda_0) \neq 0$.

$F(u, \lambda) = F^0 + F_u^0 (u - u_0) + F_\lambda^0 (\lambda - \lambda_0) + F_{uu}^0 \frac{(u - u_0)^2}{2} + F_{u\lambda}^0 (u - u_0)(\lambda - \lambda_0) + \cdots$

And let $u = u_0 + \varepsilon \phi + \varepsilon^2 u_2 + \cdots$, $\lambda = \lambda_0 + \varepsilon^2 \lambda_2 + \cdots$. Plug in, group:

$O(\varepsilon)$: $F_u^0 \phi = 0$ ✓

$O(\varepsilon^2)$: $F_u^0 u_2 + F_\lambda^0 \lambda_2 + F_{uu}^0 \phi^2 = 0$

$\Rightarrow \lambda_2 \langle \psi, F_\lambda^0 \rangle + \langle \psi, F_{uu}^0 \phi^2 \rangle = 0$ gives λ_2, $u_1 = \phi$.

Back to $\mu U = F_u(u, \lambda) U$, we expand F_u^0 to get

$\mu U = (F_u^0 + F_{u\lambda}^0 (\lambda - \lambda_0) + F_{uu}^0 (u - u_0) + \cdots) U$. Gather terms

$O(1)$: $F_u^0 \phi = 0$

2/1/23:

Hopf Bifurcation Theorem: Suppose $u_t = F(u, \lambda)$ and $F(0, \lambda) = 0 \ \forall \lambda$. Then if $F_u(0, \lambda)$ is an operator with a complex eigenvalue with real part 0, i.e. $F_u^0 \phi = i\omega_0 \phi$, then there exists a periodic branch of solutions $u(\varepsilon) = \varepsilon(\phi e^{i\omega t} + \bar{\phi} e^{-i\omega t}) + O(\varepsilon^2)$, $\lambda(\varepsilon) = \lambda_0 + O(\varepsilon^2)$, $\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + O(\varepsilon^2)$

Proof uses FAT to split the problem, then use of IFT for existence & conclusion.

2/3/23

Given polynomials $f(x) = a_n x^n + \cdots + a_0$, $g(x) = b_m x^m + \cdots + b_0$, define the <u>Sylvester Matrix</u> as $(n+m)\times(n+m)$

$$S = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & & \\ 0 & a_n & a_{n-1} & \cdots & & \\ 0 & 0 & a_n & \cdots & & \\ \vdots & \vdots & \vdots & & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & & \\ 0 & b_m & b_{m-1} & \cdots & & \\ 0 & 0 & b_m & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & b_1 & b_0 \end{pmatrix}$$

The <u>resultant</u> of f and g given by
$$R(f,g) = \det(S).$$

<u>Theorem:</u> Given $f(x) = a_n \prod_{i=1}^{n} (x-\xi_i)$, $g(x) = b_m \prod_{j=1}^{m} (x-\eta_j)$

Then
$$R(f,g) = a_n^m b_m^n \prod_i \prod_j |\xi_i - \eta_j|$$
$$= a_n^m \prod_i g(\xi_i)$$
$$= (-1)^{nm} b_m^n \prod_j f(\eta_j).$$

<u>Ex:</u> $f(x) = x^2 + ax + b$, $g(x) = f'(x) = 2x + a.$

$$S = \begin{pmatrix} 1 & a & b \\ 2 & a & 0 \\ 0 & 2 & a \end{pmatrix}$$

and $\det(S) = 4b - a^2 = 4f\left(-\frac{a}{2}\right).$

<u>2/6/23</u>

Takeaway, if working w/ polynomials the resultant is a useful tool to reduce # of variables in an equation.

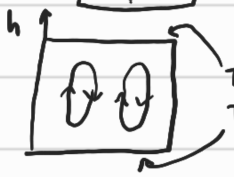
<u>Branching Examples:</u>

Pattern formation

1) Taylor vortices:
known for 100+ yrs

Formation of vortices over time.

2) Benard cells:
Temperature dependent density



gravity pulls down

T_1, T_0 } Temperatures
$T_0 > T_1$

Study temperature over time

3) Turing Patterns

<u>2/15/23</u>

For dynamics, define a flow $\phi^t : X \to X$, $t \in \mathbb{N}$ (discrete), or $\mathbb{R}$ (continuous)

An orbit is periodic if $\phi^t x_0 = \phi^{t+T} x_0$ $\forall t$ for some $T > 0$.

Invariant set $S \subseteq X$ if $x_0 \in S \Rightarrow \phi^t x_0 \in S \quad \forall t$

Stable invariant set $S_0 \subseteq U_0 \subseteq X$ if $x_0 \in U_0 \Rightarrow \phi^t x_0 \to S_0$ as $t \to \infty$.

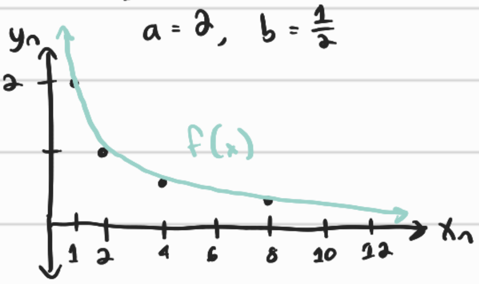
Topological Equivalence: $\phi^t : X \to X$, $\psi^t : X \to X$ are top. equiv. if $\exists\ h : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $(X = \mathbb{R}^n)$

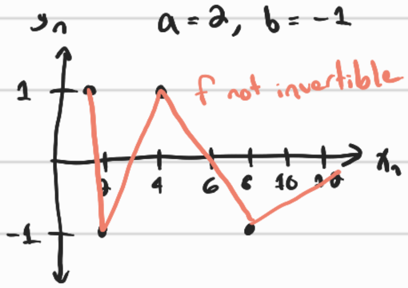
1) h is a homeomorphism (1-1 and invertible)
2) h maps orbits to orbits
3) h preserves time direction

Suppose you have DEs $x_t = f(x)$, $y_t = g(y)$, and suppose we have a map $y = h(x)$, a diffeomorphism (1-1, invertible, differentiable). Then we can get $y' = h'(x)\,x' = h'(x) f(x) = g(h(x))$. So we get the new DE $h'(x) = \frac{g(h(x))}{f(x)} = F(x,h) \in \mathbb{R}^n$, and for h to be a diffeomorphism, a soln to the DE must exist by Picard-Lindelöf.

Ex: $x \mapsto ax$, $y \mapsto by$, a discrete system.

If a, b of same sign, $\exists$ an invertible function $h(x)$ s.t. $h(x_n) = y_n$. But if $ab < 0$, then we cannot do this





Try to solve, suppose $ab > 0$. $y_n = b^n y_0 = f(a^n x_0) = f(x_n)$. Try $f(x) = kx^c$, so $f(a^n x_0) = b^n y_0 \Rightarrow k a^{cn} x_0^c = b^n y_0 \Rightarrow$ $c = \log_a(b)$, $k = y_0 / x_0^c$.

So
$$y_n = f(x_n) = \frac{y_0}{x_0^{\log_a(b)}} x_n^{\log_a(b)}.$$

We can do the same for continuous, 2D maps. For ex.

$$x \mapsto \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} x, \qquad y \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} y$$

Let $x' = ax$, $y' = by$, $a \cdot b > 0$. $x(t) = x_0 e^{at}$, $y(t) = y_0 e^{bt}$.

Now, look for $y = h(x)$.

$$y' = h'(x)\,x' = h'(x)\,ax = b\,h(x)$$
$$\Rightarrow \quad h'(x) = \frac{b}{ax} h(x)$$
$$\Rightarrow \quad h(x) = C x^{b/a}.$$

Check, $y_0 e^{bt} = y = h(x) = (C x_0 e^{at})^{b/a} = C x_0^{b/a} e^{bt} = y_0 e^{bt}$ ✓.

Note that if $a \cdot b \neq 0$, $h(x)$ is singular at $x = 0$.

<u>2/17/23</u>

$$\begin{cases} \rho_1' = \alpha \rho_1, & \theta_1' = \beta \quad \text{rotational} \\ \rho_2' = \alpha \rho_2, & \theta_2' = 0 \quad \text{line} \end{cases}$$

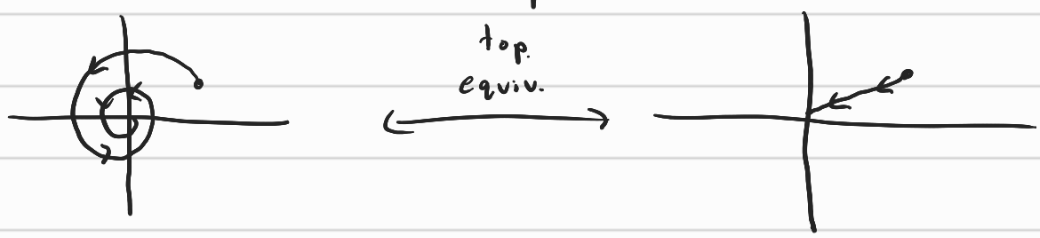
$$\rho_1 = \rho_0 e^{\alpha t}, \quad \rho_2 = \rho_0 e^{\alpha t}, \quad \theta_1 = \theta_0 + \beta t, \quad \theta_2 = \theta_0.$$

Let $t = \frac{1}{\alpha} \ln(\rho_2/\rho_0)$, then $\theta_1 = \theta_0 + \beta/\alpha \ln(\rho_2/\rho_0)$

So $\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} \longmapsto \begin{pmatrix} \rho_1 \\ \theta_1 \end{pmatrix}$ by $h\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \rho_2 \\ \theta_0 + \beta/\alpha \ln(\rho_2/\rho_0) \end{pmatrix}$

which is a continuous but not smooth map.

So



Ex: Show $y' = ay + y^2 f(y)$, $x' = ax$ are topologically equivalent where f a "nice" function.

We wish to show $y = x + x\,G(x)$, a <u>near-identity</u> <u>transformation</u> w/ $G(0) = 0$.

$$y = x + x G(x) \Rightarrow y' = (1 + G(x) + x G'(x))\,x'$$
$$= (1 + G(x) + x G'(x))\,ax$$

should equal $ay + y^2 f(y) = a(x + xG(x)) + (x + xG(x))^2 f(x + xG(x))$.

Can solve $a G' = (1+G)^2 f(x + xG)$, which is an ODE for G with IC $G(0) = 0$. Does G exist? It does if $\frac{(1+G)^2 f(x+xG)}{a}$ is Lipschitz. Well it's differentiable, so it is locally Lipschitz. And we can solve for a power series of G in terms of a power series of f using Maple.

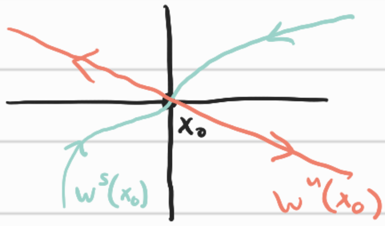
So these two systems are <u>locally topologically equivalent</u>.

Now, consider $x' = f(x)$, $x \in \mathbb{R}^n$ with $f(x_0) = 0$. We'll say the Jacobian at x_0 has n_+ eigenvalues with $\mathrm{Re}(\lambda) > 0$, n_- eigenvalues with $\mathrm{Re}(\lambda) < 0$, and no eigenvalues with $\mathrm{Re}(\lambda) = 0$.

x_0 is said to be a <u>hyperbolic point</u> if $n_0 = 0$. x_0 is said to be a <u>saddle point</u> if $n_+ \cdot n_- \neq 0$ and $n_0 = 0$.

We define the <u>stable invariant manifold</u> $W^s(x_0) := \{ x \mid \phi^t x \to x_0 \text{ as } t \to +\infty \}$.

Similarly, the <u>unstable invariant manifold</u> $W^u(x_0) := \{ x \mid \phi^t x \to x_0 \text{ as } t \to -\infty \}$.



<u>2/22/23:</u>

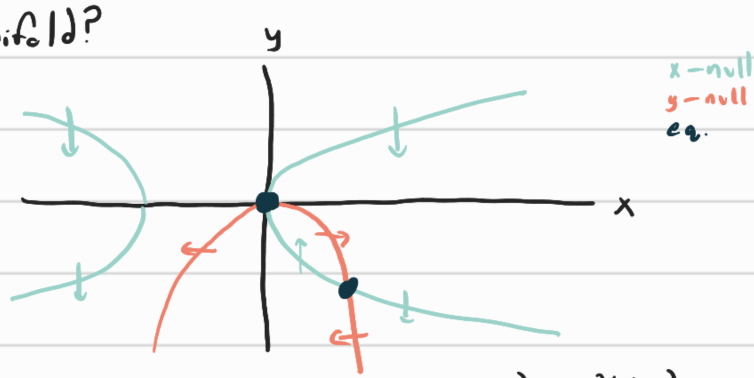
<u>Theorem:</u> Suppose $n_0 = 0$. Then given two systems $x' = f(x)$, $y' = g(y)$ are topologically equivalent around a fixed point if n_+, n_- are the same.

How do we calculate a stable/unstable manifold?

<u>Ex:</u> Consider $x' = x + \frac{1}{2}x^2 - y^2$

$y' = -\frac{1}{2}y - x^2$

We wish to know $W^s(0)$, $W^u(0)$.



Let $x' = f(x,y)$, $y' = g(x,y)$.

$y' = m'(x)x'$

For one manifold, suppose $m(x) = m_1 x + m_2 x^2 + \cdots$.

Wish to have $y = m(x)$ as a solution to the system.

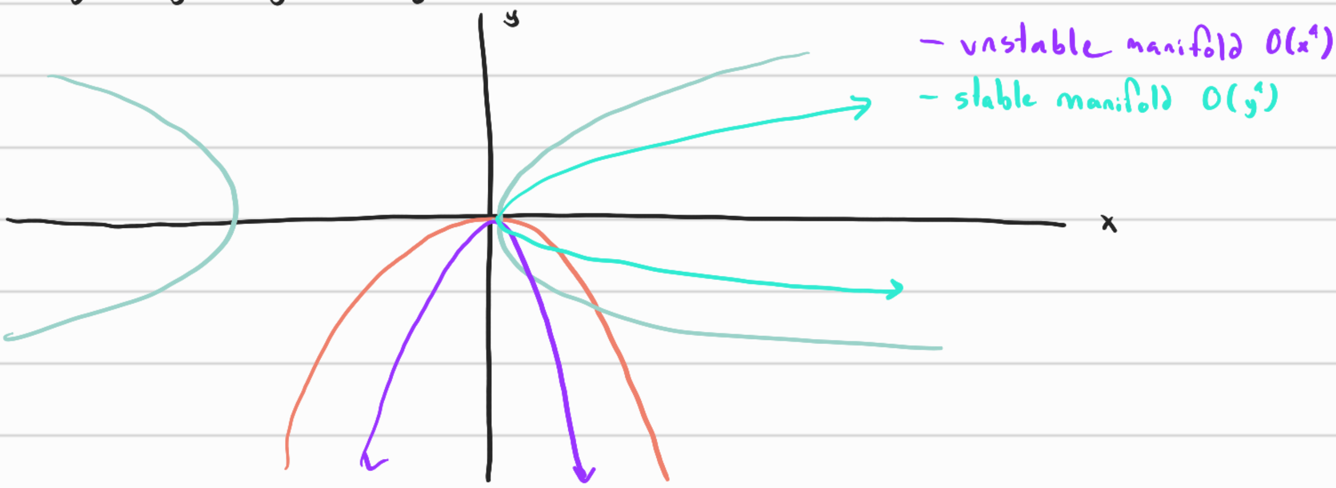
So, we want $g(x, m(x)) = y' = m'(x)x' = m'(x) f(x, m(x))$. This can give us a power series solution for $m(x)$.

Maple: $m(x) = -\frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{3}{32}x^4 + \cdots$.

We can verify that this is the unstable manifold (locally).

Similarly, let $k(y) = k_1 y + k_2 y^2 + \cdots$. Can use maple, and solve $f(k(y), y) = k'(y) h(k(y), y)$ to get that

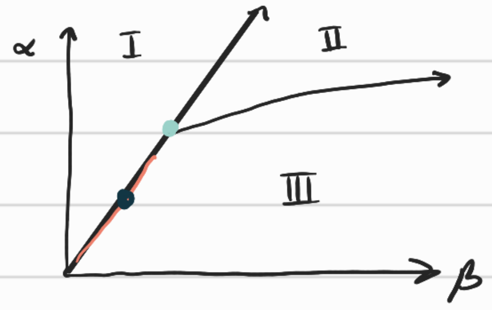
$k(y) = y^2 - y^3 + y^4 - 3y^5 + \cdots$.



2/24/23:

For a point on a boundary curve in the Bifurcation diagram, we define the <u>codimension</u> of that point to be the difference of the parameter space and the dimension of the boundary point.

<u>ex:</u>



Param-dim = 2

— -dim = 1 (line)

• -dim = 0 (point)

• has codimension $2-1=1$

• has codimension $2-0=2$

Now, given $x' = f(x, \alpha)$, $y' = g(y, \beta)$. <u>Topological equivalence</u> means there exists a parameter dependent map $h_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma : \mathbb{R}^m \to \mathbb{R}^m$ s.t. $y(t, \beta) = h(x(t, \gamma(\alpha)))$. Or $\begin{pmatrix} x(t,\alpha) \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} y(t,\beta) \\ \beta \end{pmatrix}$.

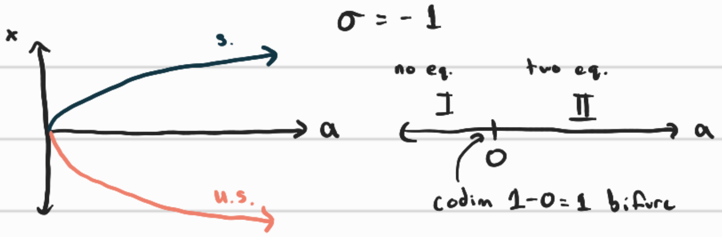
<u>Ex:</u> <u>Fold bifurcation</u>. Consider $x' = a + \sigma x^2$ where $\sigma = \pm 1$. There exist a multitude of systems equivalent to this one. We call it the <u>normal form</u> as it is the simplest such form.

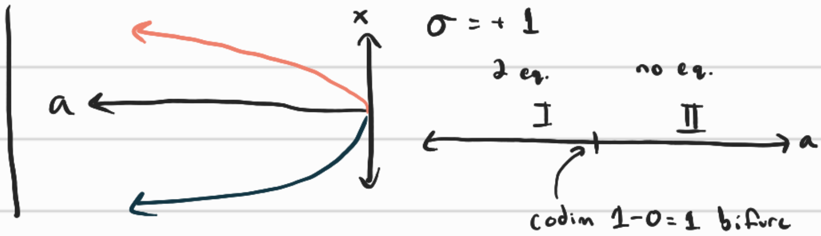
<u>Ex:</u> <u>Hopf bifurcation</u>. $z' = (\alpha + i\beta) z + \sigma z |z|^2$, $\sigma = \pm 1$, $z \in \mathbb{C}$. This is the normal form for Hopf bifurcations.

<u>Note:</u> Above 2 are <u>all</u> codimension 1 bifurcations.

<u>Theorem:</u> Consider $y' = f(y, a)$ where $\overbrace{f(0,0) = 0,\ f_y(0,0) = 0}^{\text{bifurcation conditions}}$, $\overbrace{f_{yy}(0,0) \neq 0}^{\text{non-degeneracy conditions}}$, and $\overline{f_a(0,0) \neq 0}$. Then $y' = f(y, a)$ is topologically equivalent to $x' = a + \sigma x^2$ as given above.

<u>Aside:</u> What is structure of fold bifurcation.





<u>Proof of Theorem:</u>

Idea:
1) Map parameters : Shift to match maxima of y' vs y.
2) Scale variables : Turn into parabola
3) Rescale time to normal form.

Wish to find $y = Y(a)$ so that $f_y(Y(a), a) = 0$. We know $Y(0) = 0$ as $f_y(0,0) = 0$. Will use implicit function theorem. $f_y(Y(a),a) \xrightarrow{\partial/\partial a|_{a=0}}$ $f_{yy}(0,0)Y'(0) + f_{ya}(0,0) = 0$. Since $f_{yy}(0,0) \neq 0$ by our supposition, by the implicit function theorem, $Y(a)$ exists locally around $a = 0$.

2/27/23: Proof continuation:

We know there exists $Y(a)$, solutions to $y' = f(y,a)$ around $a = 0$. Now, recall $x' = a + \sigma x^{\sigma} = F(x,a)$ $(\sigma = \pm 1)$. By what we've shown, letting $z = Y(a)$, $z' = f(Y(a),a) = b + G_2(b)z^2 + G_3(b)z^3 + \cdots =: G(z,b)$.
And now, we wish to show that $\exists$ a trasformation $z = h(x) := \alpha(b) + \beta(b)x$ such that $G(z,b) = F(x,a) = a + \sigma x^2$. We know that $\exists$ a root $F(x=r, a=-\sigma r^2) = (-\sigma r^2) + \sigma(r)^{\sigma} = 0$. So letting $z = x$, $b = -\sigma x^2$ is a point where $G(z,b) = F(x,a)$. Using implicit function theorem, we need $\frac{\partial}{\partial b} G(z(b),b) = 0$ at $b = -\sigma x^2$, $z = x$.

$$\frac{\partial}{\partial b} G(z(b),b) = \frac{\partial G}{\partial b} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial b} = (1 + G_2'(b)z^2 + G_3'(b)z^3 + \cdots) + (2G_2(b)z + 3G_3(b)z^2 + \cdots)(\alpha'(b) + \beta'(b)x)$$

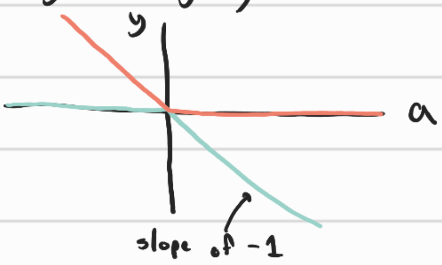
which can be solved via power series. Alternatively, we write it as so: $z' = G(z,b)$, $z = \alpha(b) + \beta(b)x$, we want $\beta(b)x' = F(x,b)$. Since $F(r, -\sigma r^2) = 0$, $F(x,b) = (b + \sigma x^2)R(x,b)$, so letting $M(x,b) = R(x,b)/\beta(b)$, we have $x' = M(b,x)(b + \sigma x^2)$, which is equal to the normal form with a scaled time (provided M positive).

3/1/23:

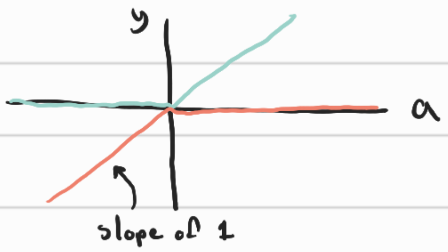
Other things we can show

1) $x' = ax + x^2 g(x,a)$, $g(0,0) \neq 0$ can be reduced to a normal form
$\Rightarrow y' = ay + \sigma y^2$, $\sigma \in \pm 1$. Transcritical bifurcation

$\sigma = 1$:



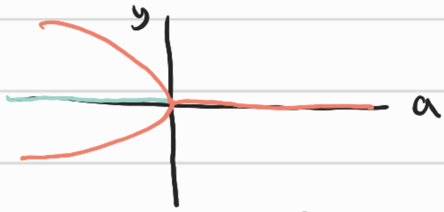
$\sigma = -1$:



2) $x' = ax + x^3 g(x,a)$, $g(0,0) \neq 0$ has a normal form of
$\Rightarrow y' = ay + \sigma y^3$, $\sigma \in \pm 1$

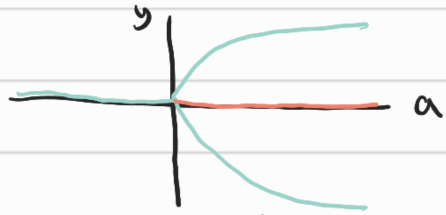
Pitchfork Bifurcation

$\sigma = 1$:



subcritical pitchfork

$\sigma = -1$:



supercritical pitchfork

What is the normal form for a Hopf Bifurcation?

Begin w/
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$
where $f(x,y)$, $g(x,y)$ have trivial linearizations.

Second, we wish to express this system w/ a complex variable. What are the eigenvalues and eigenvectors?

$$\det\begin{pmatrix} a-\lambda & -\omega \\ \omega & a-\lambda \end{pmatrix} = (a-\lambda)^2 + \omega^2 = 0 \Rightarrow \lambda = a \pm i\omega.$$

$$\begin{pmatrix} -i\omega & -\omega \\ \omega & -i\omega \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}. \quad \vec{v_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}, \; \vec{v_2} = \overline{\vec{v_1}} = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Make the transformation $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{v_1} z + \vec{v_2}\bar{z}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{v_1} z + \vec{v_2}\bar{z} \Rightarrow \begin{matrix} x = iz - i\bar{z} \\ y = z + \bar{z} \end{matrix} \Rightarrow z = \frac{y}{2} - i\frac{x}{2}$$

giving us

$$z' = \frac{\omega x + ay + g(x,y)}{2} - i\,\frac{ax - \omega y + f(x,y)}{2} = (a + i\Omega)z + h(z,\bar{z}).$$

We now wish to have a change of variables

$$z = w + h_{20}w^2 + h_{11}w\bar{w} + h_{02}\bar{w}^2 \Rightarrow w' = (a + i\Omega)w + g(w,\bar{w})$$

where $g = O(w^3)$. How to invert this to $w = w(z)$. Do this in Maple.
Will find
$$w = z - h_{20}z^2 - h_{11}z\bar{z} - h_{02}\bar{z}^2 + O(z^3).$$

Well,
$$w' = z' - 2h_{20}zz' - h_{11}z'\bar{z} - h_{11}z\bar{z}' - 2h_{02}\bar{z}\bar{z}'$$

which we plug in to the change of variables and solve s.t. all h coefficients are 0. Then $w' = (a + i\Omega)w + g(w,\bar{w})$ where g is cubic. Continue w/ another change of variables to remove cubic, and continue to remove all higer order terms.

End up w/ normal form for Hopf Bifurcation:

$$w' = \lambda w \pm \frac{1}{2} w^2 \overline{w}$$

3/3/23:

Hopf Bifurcation Theorem: Given a system of ODEs $u \in \mathbb{R}^n$, written as a Taylor's series $u' = Au + \mu Bu + O(u^2)$, where the eigenvalues of A are purely imaginary, then there exists a continuation of solutions for $\mu > 0$ or $\mu < 0$

3/13/23:

Van der Pol: $u'' - au' + u + u'u^2 + bu^3 = 0$

linearized: $u'' - au' + u = 0$

For $a < 0$, $u = 0$ is stable, for $a > 0$, $u = 0$ is unstable

Maple: 1) Set up full Van der Pol eqn

2) Let $a = \sum_{j \geq 0} a_j \varepsilon^j$

3) Let $\omega = 1 + \sum_{j \geq 1} \omega_j \varepsilon^j$, ω change of timescale

4) Let $u = \sum_{j \geq 1} u_j \varepsilon^j$, define u', u'' accordingly

5) No order 1 terms

6) $O(\varepsilon)$: $u_1'' + u_1 = 0 \Rightarrow u_1 = \alpha e^{it} + \beta e^{-it}$, note that $\alpha = \overline{\beta}$ for $u(t)$ to be real

7) $O(\varepsilon^2)$: $u_2'' + u_2 = a_1(i\alpha e^{it} - i\beta e^{-it}) + 2(\alpha e^{it} + \beta e^{it})\omega_1$. To eliminate secular terms, $a_1 = \omega_1 = 0$, and no need to include higher order periodicity, so take $u_0 = 0$

8) $O(\varepsilon^3)$ fairly complicated, subs$(t = \log(q)/I, eq3)$ changes $e^{it} \to q$ and $e^{-it} \to \frac{1}{q}$, $e^{2it} = q^2$. We then have to remove terms of the forms q, q^{-1} to avoid secular terms. Also eventually find $\alpha = c + 0i$, so $\beta = c - 0i = c$.

9) $u = 2c\sin(t) + \varepsilon^2 c^2 + O(\varepsilon^3) \Rightarrow$ Hopf Bifurcation to right

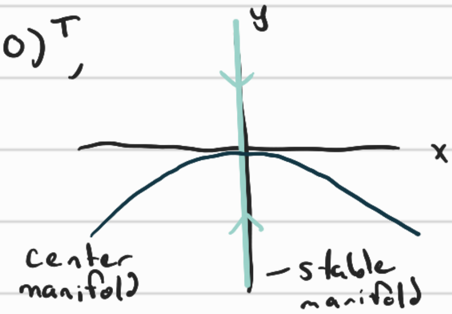
We know what the stable and unstable manifolds are. What about the eigenspaces associated with $\text{Re}(\lambda) = 0$? If $n_0 > 0$.

The <u>center manifold</u> is an invariant manifold locally tangent to the eigenspace of $\mathrm{Re}(\lambda) = 0$.

<u>Ex:</u> $x' = xy + x^3$, $y' = -y - 2x^2$, $\quad J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad n_+ = 0$, $n_0 = 1$, $n_- = -1$

<u>3/15/23:</u> (continue above example)

Note that if $y = 0$, $x' = x^3$, suggesting the origin is unstable (incorrect). So $y = 0$ is not the center manifold. We know that the center manifold must be locally tangent to $(1,0)^T$,



so have $Y(x) = a_2 x^2 + a_3 x^3 + \cdots$

To satisfy the ODE,

$$(-2-a_2)x^2 - a_3x^3 - \cdots = -Y(x) - 2x^2 = y' = Y'(x)\,x'$$

$$= \left(2a_2 x + 3a_3 x^2 + \cdots\right)\left(x(a_2x^2 + a_3x^3 + \cdots) + x^3\right)$$

$$= (2a_2^2 + a_2)x^3 + (2a_2a_3 + 3a_2a_3 + 3a_3)x^4 + O(x^5)$$

So $-2 - a_2 = 0 \Rightarrow a_2 = -2$,

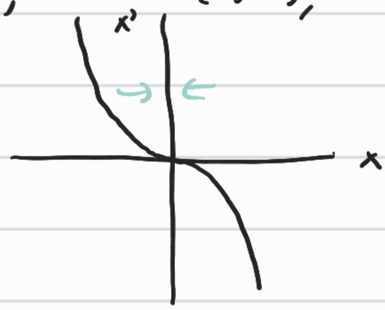
$-a_3 = 2a_2^2 + a_2 = 8 - 2 = 6 \Rightarrow a_3 = -6$

$-a_4 = 2a_2a_3 + 3a_2a_3 + 3a_3 = 24 + 36 - 18 = 42 \Rightarrow a_4 = -42$.

I.e., the center manifold has the form (<u>locally</u>)

$$Y(x) = -2x^2 - 6x^3 - 42x^4 + O(x^5)$$

And on this manifold, near $(0,0)$, $x' = -x^3 + O(x^4)$



So while the Jacobian approximated local instability, it is in fact locally stable.

In general: $u' = \frac{1}{2}\sigma u^2 + u\langle b, v\rangle + \frac{1}{6}\delta u^3$, $u \in \mathbb{R}$, $v, b \in \mathbb{R}^k$

$v' = Cv + \frac{1}{2}au^2 + \cdots$, $C \in \mathbb{R}^{k\times k}$, invertible.

Center manifold like $V(u) = a_2 u^2 + a_3 u^3 + \cdots$, $a_2, a_3, \cdots \in \mathbb{R}^k$.

$$V'(u)u' = V'(u)\left(\frac{1}{2}\sigma u^2 + u\langle b, V(u)\rangle + \frac{1}{6}\delta u^3 + \cdots\right)$$
$$= v' = CV(u) + \frac{1}{2}au^2$$

Can use ODE theory to see if soln exists, and power series methods to find a local solution.

Power series tells us

- $Ca_2 + \frac{1}{2}a = 0 \Rightarrow a_2 = \frac{-1}{2}C^{-1}a$
- this tells us nature of u' locally
- can continue

What if there is a parameter?

Ex: $x' = \alpha + xy + x^3$

$y' = -y - 2x^2$

$\alpha' = 0$

The secret to this problem is to include an additional parameter. Then search for manifold $Y = f(x, \alpha)$.

Gives us the Jacobian $J(0,0,0) = \begin{array}{c} x \\ \alpha \\ y \end{array}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\\0\end{pmatrix}$ associated w/ $\lambda = 0$

$$Y(x,\alpha) = a_1 x\alpha + a_2 x^2 + a_3\alpha^2 + \cdots$$

Then set up $y' = Y_x(x,\alpha)x' + Y_\alpha(x,\alpha)\alpha' = Y_x(x,\alpha)x'$

$$a_1 x\alpha + a_2 x^2 + a_3\alpha^2 + \cdots = (a_1\alpha + 2a_2 x + \cdots)(\alpha + xY(x) + x^3)$$

and solve in say Maple.

Consider $x' = Ax + F(x) \in \mathbb{R}^n$. And $Aq = 0$, $A^T p = 0$, $\langle q, p\rangle = 1$.

Let $u = \langle p, x\rangle$, and $y = x - \langle p, x\rangle q \Rightarrow x = uq + y$.

So $u' = \langle p, x'\rangle = \langle p, Ax\rangle + \langle p, F(x)\rangle$

$$= \langle p, Ay\rangle + \langle p, F(uq + y)\rangle.$$

Then $y' = Ay - F(uq + y) - \langle p, F(uq + y)\rangle$

3/17/23:

Ex: $u''' + au'' + bu' + u - u^2 = 0$.

There exist 2 steady states: $u \equiv 0$, and $u \equiv 1$.

Linearizing about $u = 0$, $u''' + au'' + bu' + u = 0$.

So looking for e.v.s, $\lambda^3 + a\lambda^2 + b\lambda + 1 = 0$,

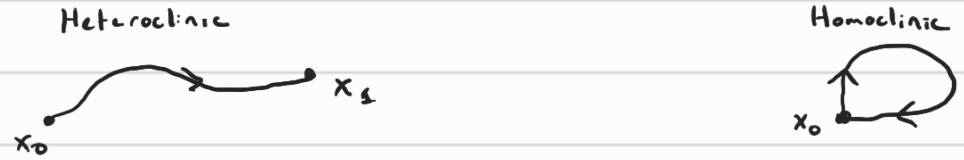
let $\lambda = i\omega$: $-i\omega^3 - a\omega^2 + ib\omega + 1 = 0$

$\Rightarrow b = \omega^2$, $a = \frac{1}{\omega^2}$, so $ab = 1$.

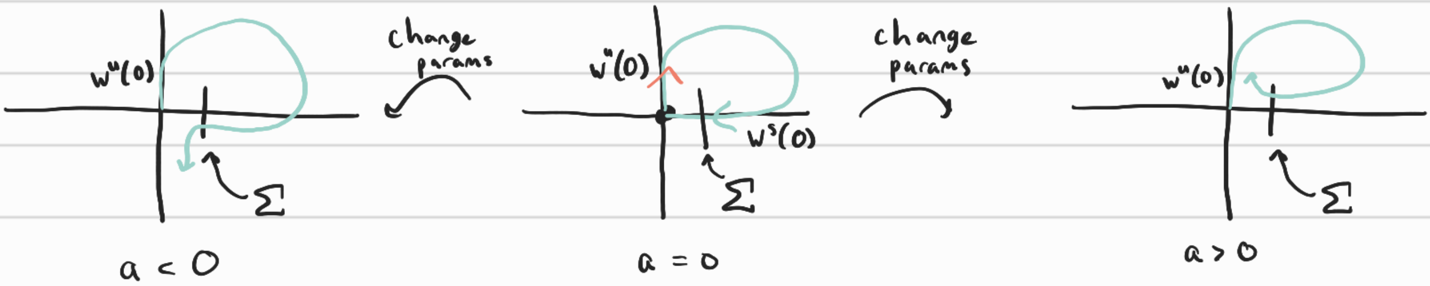
We then change time, $\omega t = \tau$, set $b = c = \omega^2$, set $a = \frac{1}{\omega^2}(1 + \delta) = \frac{1}{c}(1 + \delta)$, to calculate perturbations in a.

3/20/23: Homoclinic Bifurcations

A homoclinic connection is one s.t. $\phi^t x \to x_0$ as $t \to \pm\infty$. A heteroclinic connection is one s.t. $\phi^t x \to x_0$ as $t \to -\infty$, and $\phi^t x \to x_1$ as $t \to \infty$.



In 2-dimensions, a homoclinic orbit is structurally unstable.



Define $\beta = \xi(a)$ to be the y coord where the flow intersects Σ. This is called a split function. We wish to turn this into a Poincaré map.

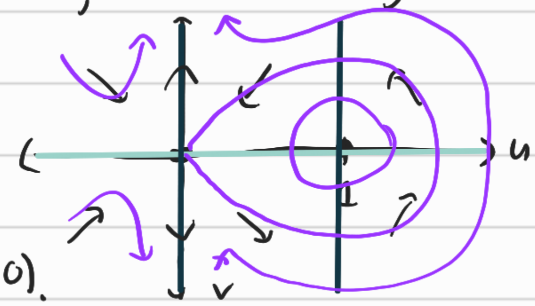
3/24/23

Hopf Bifurcation: $Re(\lambda) = 0$, there must be a branch of periodic solns.

Global Hopf Thm: Hopf curves cannot simply end, they can either go to another Hopf point, or the solution becomes unbounded in its amplitude or period

Ex: $u'' + u(u-1) + au' + bu^3u' = 0$, a and b "small".

First, suppose $a=b=0$, then $u'' + u(u-1) = 0$, which letting $v=u'$ yields $u' = v$, $v' = -u(u-1)$. Thru linearizing, $u=0$ a saddle, $u=1$ a center.



Perturbing a changes the stability of $(1,0)$.

Introduce small scaling ε s.t. $a \mapsto \varepsilon a$, $b \mapsto \varepsilon b$, wish to perturb homoclinic orbit. Let $u = u_0 + \varepsilon u_1 + \cdots$, u_0 is the homoclinic orbit satisfying $u_0'' + u_0(u_0 - 1) = 0$. Now,

$O(\varepsilon)$: $u_1'' + (2u_0 - 1)u_1 + au_0' + bu_0^3 u_0'$.

Where we have a linearized operator $Lu = u'' + (2u_0(x) - 1)u$. Need to solve $Lu = -au_0 - bu_0^3 u_0'$, use Fredholm Alternative Theorem.

Working in $L^2(-\infty, \infty)$, L a Sturm-Liouville operator, what is the nullspace?

Can show u_0' is in L's nullspace. (Let $u_0 = u_0(x+\phi)$ as solutions of homoclines can be shifted)

$\frac{\partial}{\partial \phi}$: $u'' + u^2 - u = 0 \Rightarrow u_\phi'' + 2uu_\phi - u_\phi = 0$

$\Rightarrow u_\phi'' + (2u-1)u_\phi = 0$

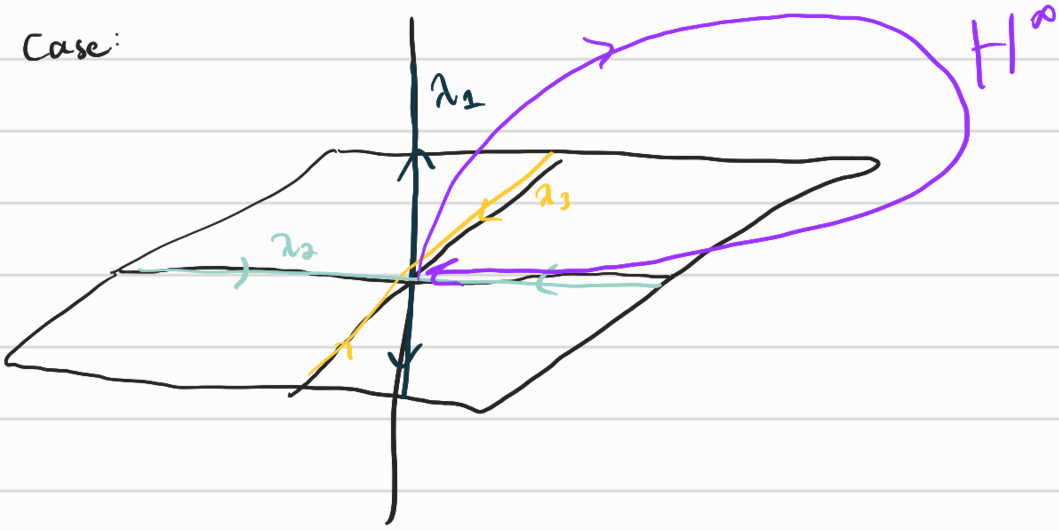
$\Rightarrow Lu_\phi = 0$.

Keener: Solution to $u_0(t) = \frac{3}{2}\text{sech}^2(t/2)$. In Maple, set up right hand side as $R := -au_0 - bu_0^3 u_0'$. Since u_0' in the nullspace, set up in maple as int(R * u_0', t=-infinity..infinity) which must be zero, yields $\frac{432b}{385} + \frac{6a}{5}$, setting this equal to zero yields curve in (a,b) space where $\exists$ homoclines.

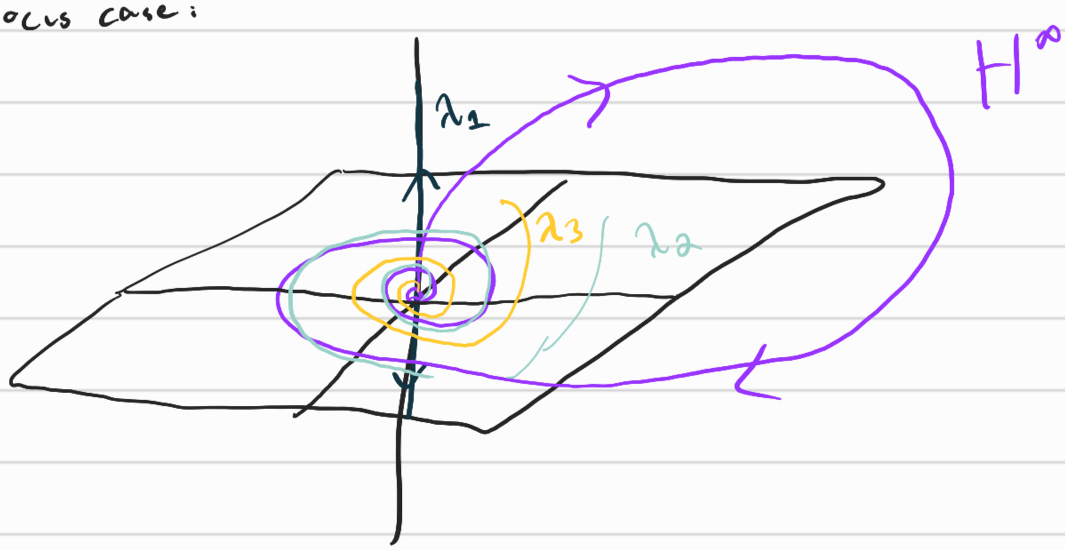
3/27/23:

Consider $x' = f(x,a)$, with $x \in \mathbb{R}^3$, $f(x_0, a) = 0$ $\forall a$, with x_0 hyperbolic, a saddle point. We assume $n_+ = 1$, $n_- = 2$. If not, we reverse time. WLOG, consider $\text{Re}(\lambda_3) < \text{Re}(\lambda_2) < 0 < \text{Re}(\lambda_1)$. So in the stable manifold, λ_3 eigenspace is "faster", so λ_2 the principal eigenvalue as values get pushed quickly to λ_2 eigenspace. Now, assume/consider a homoclinic orbit, H^∞, see below

Saddle case:

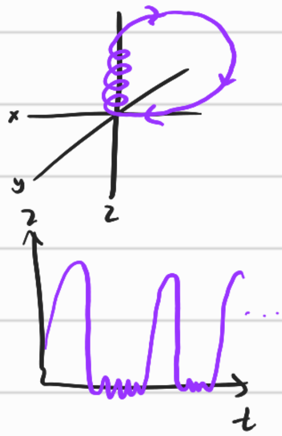


Saddle-focus case:



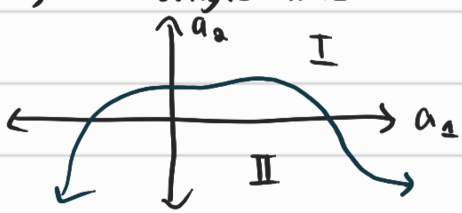
Define the <u>saddle quantity</u> $\sigma_0 = Re(\lambda_1) + Re(\lambda_2)$. Recall the splitting parameter β. Now, perturb param a.

	saddle $(Im(\lambda_i) = 0)$	saddle-focus
$\sigma_0 > 0$	unique stable periodic orbit w/ $\beta > 0$	unique stable periodic orbit w/ $\beta > 0$
$\sigma_0 < 0$	unique saddle cycle (cycle is not "fully" attracting)	infinite number of saddle cycles



<u>3/29/23</u>:

We will now discuss codimension 2 bifurcations. First, recall <u>codimension 1</u> means that in parameter space, a single line divides behavior



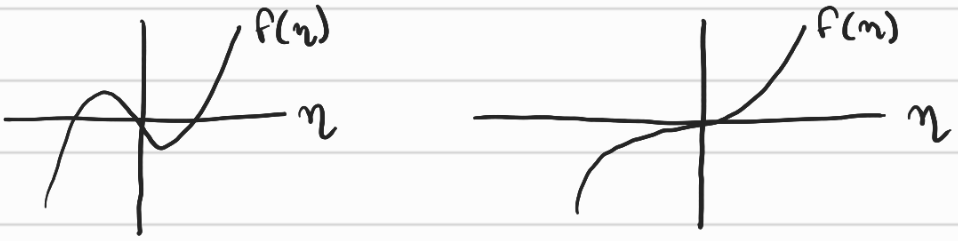
> Ex: Fold bifurcation $x' = a + \sigma x^2$. Only one equilibria for $a = 0$, two for $\text{sign}(a) \neq \text{sign}(\sigma)$, zero for $\text{sign}(a) = \text{sign}(\sigma) \neq 0$.

Cusp Bifurcation: $x' = f(x, \alpha)$, $x \in \mathbb{R}^1$, $\alpha \in \mathbb{R}^2$, and normalize s.t. $f(0,0) = 0$. And assume that $f_x(0,0) = 0$, $f_{xx}(0,0) = 0$, and $f_{xxx}(0,0) \neq 0$. So about $x = 0$,

$$f(x,\alpha) = 0 + 0x + 0x^2 + f_3(\alpha)x^3 + O(x^4).$$

You can show, $\exists$ a smooth change of variables to arrive at the normal form for the cusp bifurcation.

$$\eta' = \beta_1 + \beta_2 \eta + \sigma \eta^3, \quad \sigma = \pm 1$$



$\beta_1 = 0,\ \beta_2 < 0$ $\qquad$ $\beta_1 = 0,\ \beta_2 \geq 0$

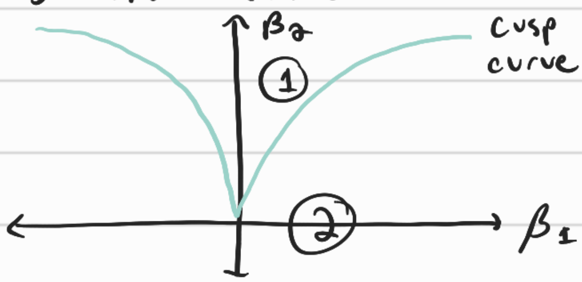
and perturbing β_1 shifts the curve upwards and downwards.
Roots occur where $\beta_1 + \beta_2 \eta + \sigma \eta^3 = 0$,
double roots when $\beta_2 + 3\sigma\eta^2 = 0$.
Using the resultant, simultaneous roots when $\sigma^2(4\beta_2^3 - 27\beta_1^2) = 0$.
This curve has a cusp due to cube and squares.

3/31/23:

Now, consider $\mathbb{R}^n$. Consider origin is an equilibrium point with $n_0 = 1$, and $n_- + n_+ = n - 1$. The normal form looks like:

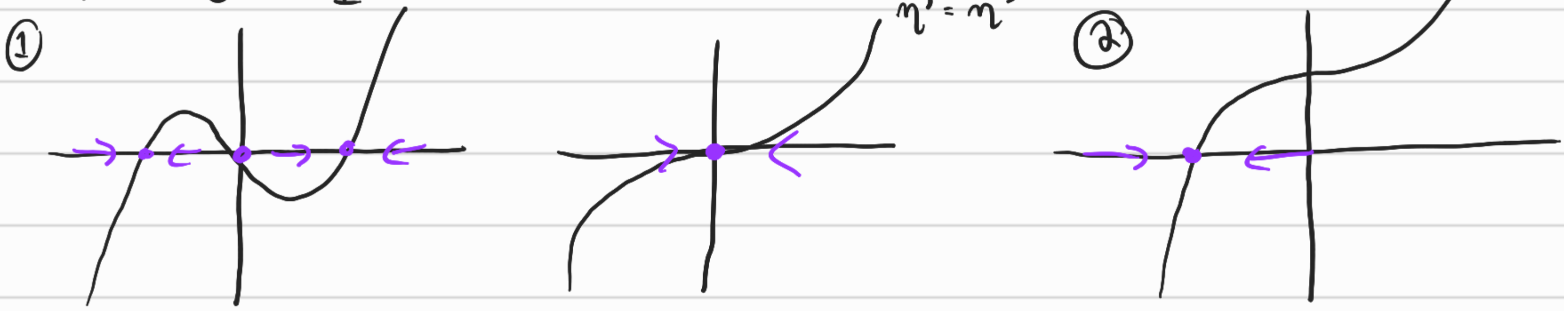
$$\begin{cases} \eta' = \beta_1 + \beta_2 \eta \pm \eta^3 & \text{along center manifold} \\ \xi_-' = -\xi_- & \text{along stable manifold} \\ \xi_+' = \xi_+ & \text{along unstable manifold} \end{cases}$$

Recall, cusp is $4\beta_2^3 - 27\beta_1^2 = 0$



At the origin, $\beta_1 = \beta_2 = 0$, $\eta' = \pm \eta^3$, a degenerate repeated, single equilibrium point at $\eta = 0$. Not structurally stable. In region ①, 3 steady states, origin changes stability. In region 2, one equilibrium point, same stability as original origin.

Ex: $\sigma = +1$



Now, consider the Bautin bifurcation, a degenerate Hopf bifurcation. Consider $z' = (\lambda + i)z + l_1(a) z|z|^2$.

What happens when $l_1(a) = 0$? Must go to higher order

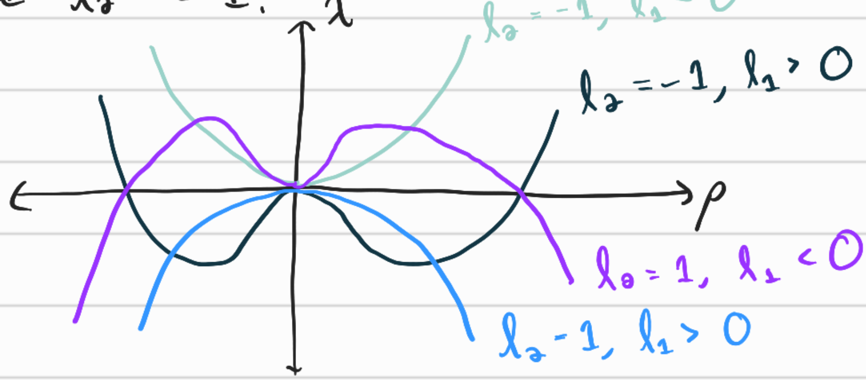
$$z' = (\lambda + i) z + l_1(a) z|z|^2 + l_2 z|z|^4.$$

We convert to polar coordinates, let $z = \rho e^{i\theta}$. Then $z' = \rho' e^{i\theta} + i\theta' \rho e^{i\theta}$. Solving for ρ':

$$\rho' = \lambda\rho + l_1\rho^3 + l_2\rho^5.$$

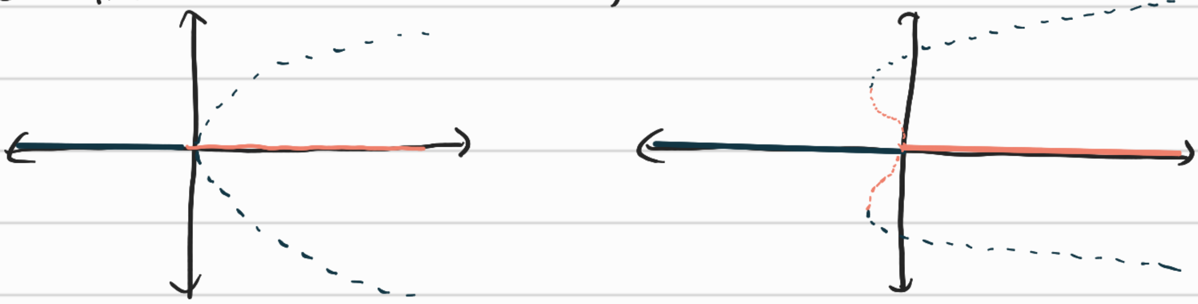
Solve for equilibria: $\rho = 0$, $\lambda = -l_1\rho^2 - l_2\rho^4$.

WLOG, can take $l_2 = \pm 1$.



We see there is either 0 or 2 equilibria (real). Flipping axes reveals periodic solutions, Hopf curves.

Fixing $l_2 = -1$, $\lambda_1 < 0$ incurs a supercritical Hopf. Then $\lambda_1 > 0$ incurs a SNP, subcritical, turns to saddle node periodic, to stable periodic.



$\lambda_1 < 0$ $\lambda_1 > 0$

Now, we will talk about the <u>Takens-Bogdanov Bifurcation</u>.

Consider a linearization of a 2D system with linearization

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ Note: 0 is a double eigenvalue.

Consider a 2D center manifold (can have $\mathbb{R}^n$ with stable & unstable)

$$\left.\begin{array}{ll} \text{Takens system:} & \begin{aligned} \eta_1' &= \eta_2 + \beta_2\eta_1 + \eta_1^2 \\ \eta_2' &= \beta_1 + s\eta_1^2 \end{aligned} \\ \\ \text{Bogdanov system:} & \begin{aligned} \eta_1' &= \eta_2 \\ \eta_2' &= \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 \end{aligned} \end{array}\right\} \quad s \in \pm 1.$$

Takens system has steady states at $\eta_1 = \pm\sqrt{\frac{\beta_1}{s}}$, 2, 1, or 0 based on sign of s and value of β_1.

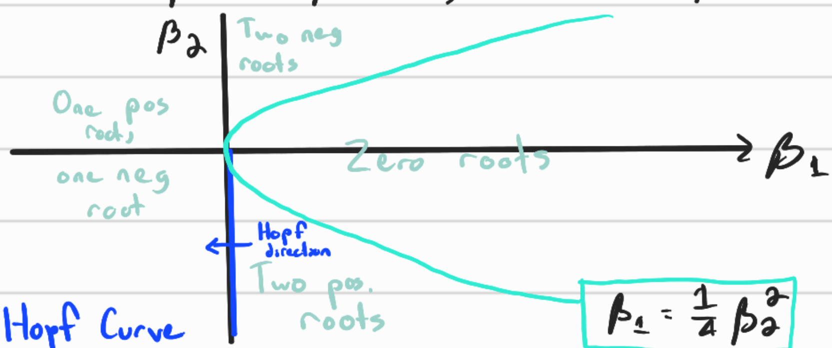
Bogdanov system has steady states at $\eta_2 = 0$, $\beta_1 + \beta_2\eta_1 + \eta_1^2 = 0$, again, 2, 1, or 0. In both cases, fold bifurcations.

We will take s to be -1 WLOG, as otherwise can reverse time, shift params.

Look at Bogdanov equation.

$$\beta_1 + \beta_2\eta_1 + \eta_1^2 = 0 \Rightarrow \eta_1 = \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1}}{2}$$

So 2 solns if $\beta_2^2 - 4\beta_1 > 0$, 1 if $\beta_2^2 - 4\beta_1 = 0$, 0 otherwise.



Can show analytically when $\beta_2 < 0$, crossing $\beta_1 = 0$ results in a Hopf Bifurcation.

4/5/23: Can compute (still taking $s = -1$)

$$J(\eta_1, \eta_2) = \begin{pmatrix} 0 & 1 \\ \beta_2 + 2\eta_1 + s\eta_2 & -\eta_1 \end{pmatrix}.$$

At $(0,0)$, characteristic eqn is $\lambda^2 - \beta_2 = 0 \Rightarrow \lambda = \pm\sqrt{\beta_2}$, purely imaginary when $\beta_2 < 0$, $\beta_1 = 0$. Steady state always at $\eta_2 = 0$, so, characteristic eqn otherwise

$$(-\lambda)(-\eta_1 - \lambda) - (\beta_2 + 2\eta_1) = 0$$

$$\Rightarrow \lambda^2 + \eta_1\lambda - (\beta_2 + 2\eta_1) = 0$$

$$\Rightarrow \lambda = \frac{\eta_1 \pm \sqrt{\eta_1^2 + 4(\beta_2 + 2\eta_1)}}{2(\beta_2 + 2\eta_1)}$$

$$= \frac{\eta_1 \pm \sqrt{\eta_1^2 + 8\eta_1 + 4\beta_2}}{2(\beta_2 + 2\eta_1)}$$

which has real part if $\eta_1 \neq 0$. $\eta_1, \beta_2 > 0 \Rightarrow$ real opposite sign. When $\eta_1 < 0$, real part $\to \pm\infty$ when $\beta_2 \to -2\eta_1 > 0$. Real part positive when $\beta_2 < -2\eta_1 = \beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1} \Rightarrow 0 < \pm\sqrt{\beta_2^2 - 4\beta_1}$, so one stable, one unstable when 2 equilibria.

In Maple, set up eqn, set $a = \beta_1$, $\overbrace{b^2 = -\beta_2}^{\text{only search } \beta_2 \leq 0.}$, and scale time $t \mapsto \omega t$. Then, fix b, vary a to find periodic solutions. Can solve that the Hopf opens to the left.

4/7/23:

We now wish to find the homoclinic bifurcation where the set of periodic solutions dissapear. Recall

$$\begin{cases} \eta_1' = \eta_2 \\ \eta_2' = \beta_1 + \beta_2\eta_1 + \eta_1^2 - \eta_1\eta_2. \end{cases} \qquad \text{Takens-Bogdanov}$$

Define $\eta^{\pm} := \frac{-\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1}}{2\beta_1}$, and perform the change of variables $\xi_1 = \eta_1 - \eta^-$, $\xi_2 = \eta_2$.

Then we get the system

$$\xi_1' = \xi_2,$$
$$\xi_2' = \beta_1 + \beta_2(\xi_1 + \eta^-) + (\xi_1 + \eta^-)^2 - (\xi_1 + \eta^-)\xi_2.$$

The system is shifted so always has root at 0. The other root is $r = \sqrt{\beta_2^2 - 4\beta_1}$. Now, rescale so other root is at 1, let $\xi_1 = rx$, $\xi_2 = \gamma y$, $t = s\tau$,

$$\frac{dx}{d\tau} = s\frac{dx}{dt} = \frac{s\gamma}{r}y, \quad \text{we'll choose } \frac{s\gamma}{r} = 1$$

$$\frac{dy}{d\tau} = s\frac{dy}{dt} = \frac{s}{\gamma}\frac{d\xi_2}{dt} = \frac{s}{\gamma}\left(r^2(x-1)x + \gamma y(rx + \eta^-)\right)$$

So, choosing s, γ properly, we get

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = (x-1)x - sy(rx + \eta^-).$$

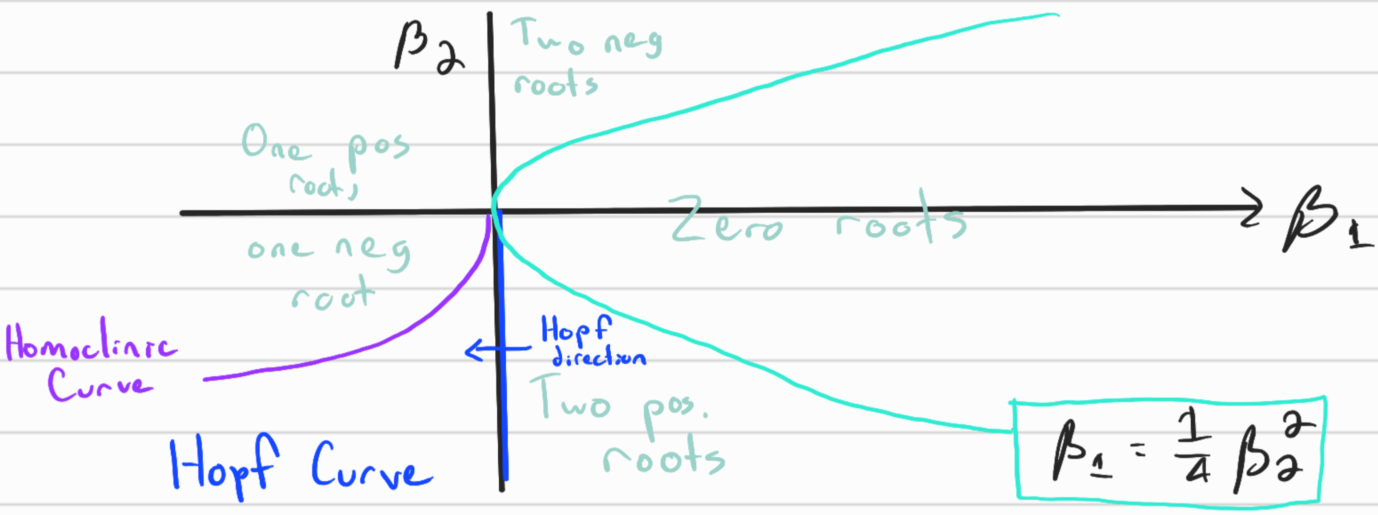
Now, how do we find the homoclinic orbit? Let $\varepsilon_1 = sr$, and $\varepsilon_2 = s\eta^-$. Can rewrite as:

$$x'' + (1-x)x + x'(\varepsilon_1 x + \varepsilon_2) = 0.$$

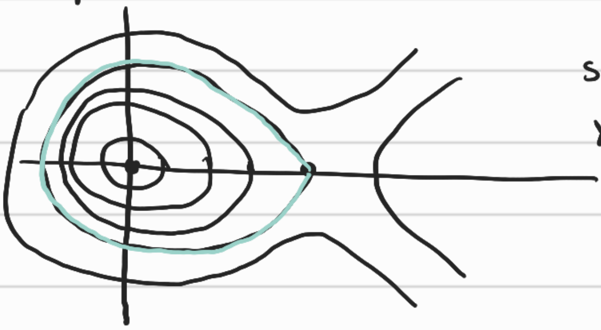
Let $\varepsilon_1 = \varepsilon a_1$, $\varepsilon_2 = \varepsilon a_2$, invoke FAT and find that solutions exist if $a_1 + 7a_2 = 0$. Converting back,

$$r + 7\eta^- = 0 \Rightarrow \frac{5}{2}r + \frac{7}{2}\beta_2 = 0 \Rightarrow 5\sqrt{\beta_2^2 - 4\beta_1} + 7\beta_2 = 0$$

$\Rightarrow \beta_1 = \frac{-6}{25}\beta_2^2$, the homoclinic curve in β_1, β_2 space.

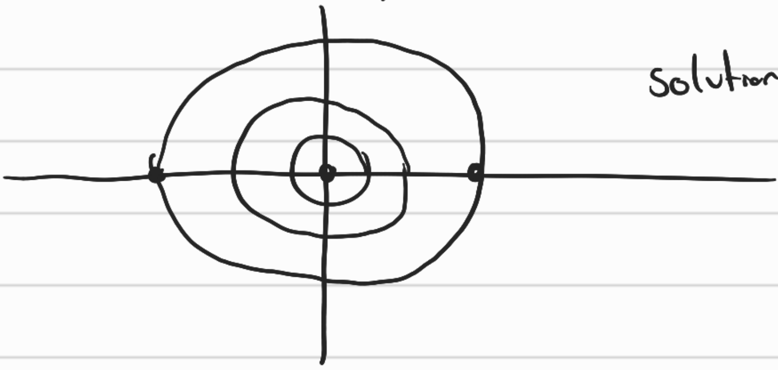


Reminder, we had the base problem $x'' + x(1-x) = 0$ which has phase plane. We then perturbed it in a special way to preserve the homoclinic orbit.



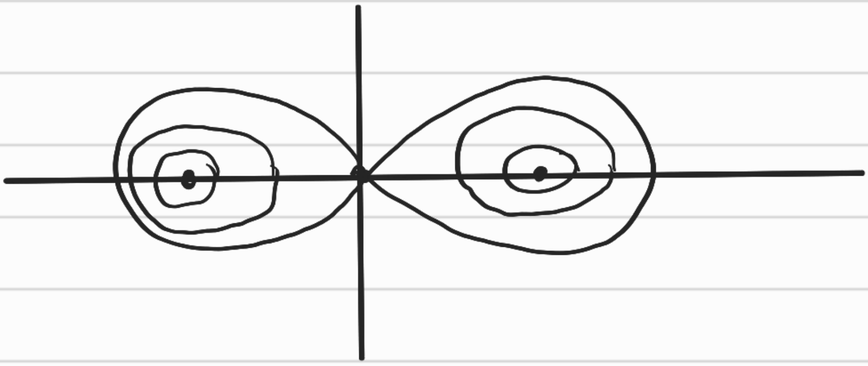
solutions like $x(t) = \tanh^2(t)$.

Heteroclinic orbits can show up w/ $x'' + x(1-x^2) = 0$



solutions like $x(t) = \tanh(t)$

Similarly for $x'' - x(1-x^2)$

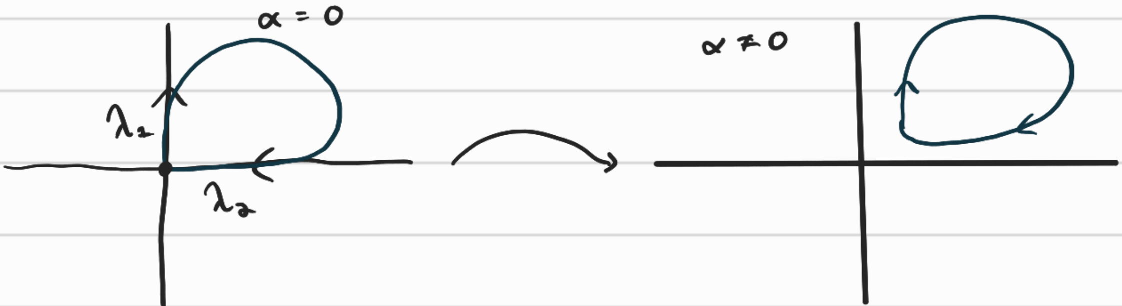


4/10/23:

Digression into homoclinic bifurcation:

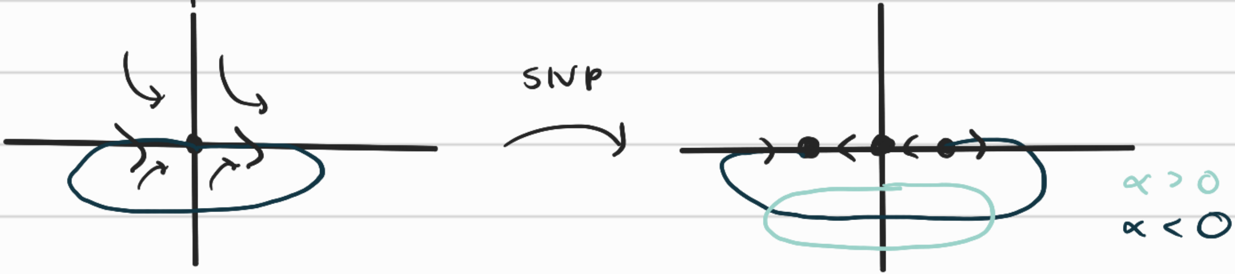
Recall saddle quantity $\sigma = \lambda_1 + \lambda_2$ for a saddle point with a homoclinic orbit.

1)



Stable if $\sigma < 0$, unstable if $\sigma > 0$

2)



4/12/23:

Ex: Bazykin Equations: Predator-prey model

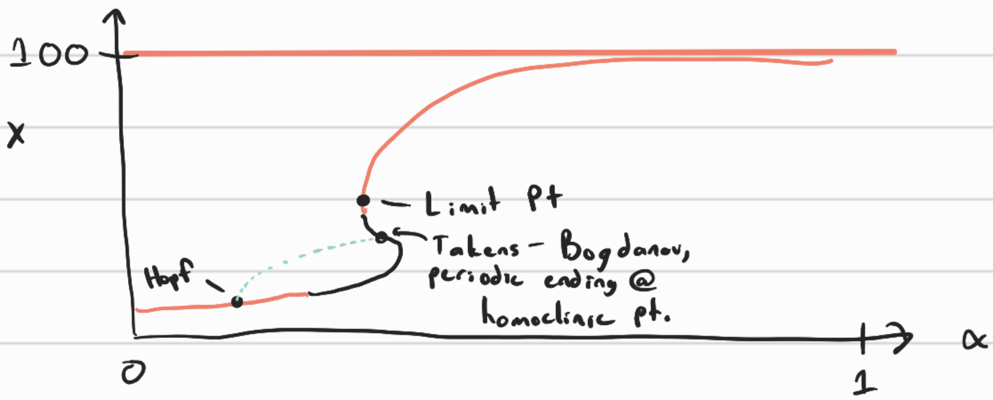
$$x' = x - \frac{xy}{1+\alpha x} - \varepsilon x^2 \quad \text{— Prey}$$

$$y' = -\gamma y + \frac{xy}{1+\alpha x} - \delta y^2 \quad \text{— Predator}$$

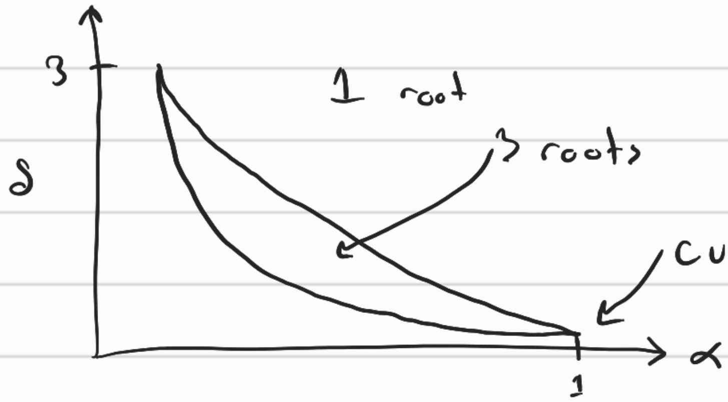
Note: $x - \varepsilon x^2$ yields logistic growth in x, $\frac{xy}{1+\alpha x}$ yields predation, denominator says predation saturates for large x.
$-\gamma y - \delta y^2 = -\gamma(1+\delta y)$ yields

Steady states @ $(0,0)$, $(\frac{1}{\varepsilon}, 0)$, and (x^*, y^*) w/ both positive ($\exists$ a negative y steady state too, ignore). Now, we'll look for bifurcations.

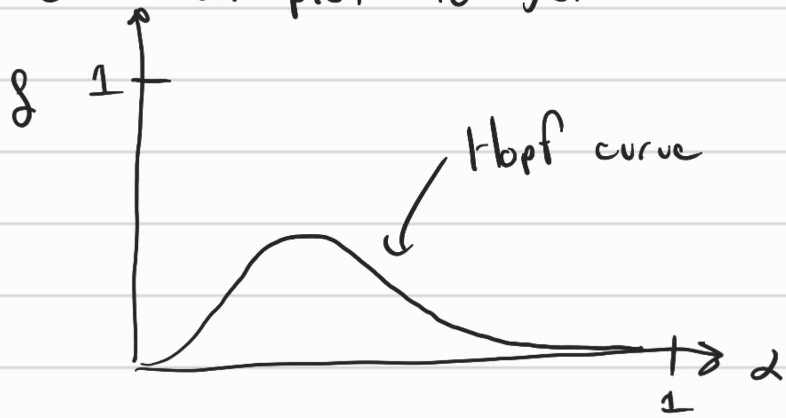
XPP: Initialize $\alpha = 0.05$, $\delta = 0.17$, $\gamma = 1.0$, $\varepsilon = 0.01$, $x(0) = y(0) = 1$.



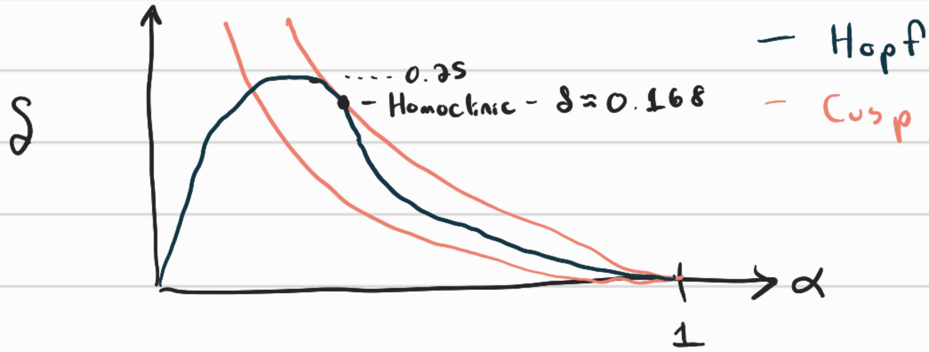
Maple: Find Jacobian, set trace $= 0$, need determinant > 0 along steady states by $y = \text{solve}(\partial x, y)$, and factor$(\partial x)$ which reveals a cubic in x and $x = \frac{1}{\varepsilon}$. Find repeated roots of the cubic thru the resultant. We substitute $\gamma = 1$, and let $\varepsilon = 0.01$, then you get an expession in terms of α, δ.
Use a contourplot



Set $nTr = \text{numer}(\text{Trace})$, then we take the resultant of nTr, and cubic polynomial. Again, substitute $\gamma = 1$, $\varepsilon = 0.01$, and take a contour plot to get



Keener copied both into Matlab, put both on same plot



In XPP, increasing $\delta = 0.22$ results in two Hopf points, curves go from one to another.