PDEs Review

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1 Laplace and Poisson Equations

We first seek a similarity solution to the Laplace equation

$$-\Delta u = 0 \quad x \in \mathbb{R}^n$$

which is rotation invariant, translation invariant, and satisfies the proper scaling. Seeking a solution of the form $u(x) = v(|x|) = v(r)$, and plugging into the PDE, we find solutions of the form

$$u(x) = v(r) = \begin{cases} A\log(r) + B & n = 2 \\ Ar^{2-n} + B & n > 2. \end{cases}$$

The **fundamental solution** to the Laplace equation is given by

$$\Phi(x) = \begin{cases} \frac{-1}{2\pi}\log(|x|) & n = 2 \\ \frac{1}{n(n-2)\alpha_n|x|^{n-2}} & n > 2, \end{cases}$$

where $\alpha_n = |B(0,1)|$ is the volume of the n-ball, and $n\alpha_n = |\partial B(0,1)|$ is its surface area. Can prove that from the chosen scaling, $-\Delta\Phi$ equals the dirac delta function centered at zero, δ_0.

The **convolution** of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ (provided integrability conditions) is a new function $(f * g) : \mathbb{R}^n \to \mathbb{R}$ given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Properties of convolutions:

1. Symmetry: $(f * g) = (g * f)$
2. Differentiability: $\partial_{x_i}(f * g) = (\partial_{x_i} f * g)$
3. Smoothing: If g is $C^k(\mathbb{R}^n)$, then $(f * g)$ is $C^k(\mathbb{R}^n)$ regardless of the differentiability of f

Theorem 1. Given $f \in C_c^2(\mathbb{R}^n)$, then $-\Delta(\Phi * f) = f$, hence, $u = (\Phi * f)$ solves the Poisson equation $-\Delta u = f$.

Theorem 2. If u is **harmonic**, i.e. $-\Delta u = 0$, in a domain U, and $\overline{B(x,r)} \subset U$, then u satisfies the **mean value property**:

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS = \frac{1}{|B(x,r)|} \int_{B(x,r)} u dy.$$

This proof is done by defining

$$\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS,$$

performing a change of variables $y \in \partial B(x,r) \to z \in \partial B(0,1)$ so that

$$\phi(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} u(x + rz) dS(z),$$

differentiating, and using divergence theorem to prove that $\phi'(r) = 0$ for not too large r.

Corrolary. If $u \in C^2(U)$ such that for all $x \in U$, there exists $r > 0$ such that

$$u(x) = \frac{1}{|\partial B(x,\rho)|} \int_{\partial B(x,\rho)} u dS$$

for all $\rho \in (0,r)$, then u is harmonic.

Theorem 3. Suppose that $u \in C(U)$ satisfies the mean value property on U. Then, $u \in C^\infty(U)$ and u is harmonic on U.

Proof is performed by mollifying u with a rescaled but smooth bump function of the form

$$\eta(x) = \begin{cases} C_n \exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

where C_n is a constant only dependent on the dimension n so that η integrates to 1. The defining $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$, we have that η_ϵ is C^∞ for all $\epsilon > 0$, and approaches the dirac delta as $\epsilon \to 0$. Can then prove that $u_\epsilon = (u * \eta_\epsilon)$ is equal to u on U^ϵ which is the original domain U except for points within ϵ of the boundary.

$$\text{So, } -\Delta u = 0 \iff \text{Mean value property} \implies u \in C^\infty.$$

Theorem 4. Suppose that $u \in C(\overline{U})$ is harmonic in U. Then, the **weak maximum principle** holds:

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Additionally, the **strong maximum principle** holds: If U is connected and there exists $x_0 \in U$ such that x_0 maximizes u on $\overline{U}$, then u is constant on U.

The strong maximum principle is proved by showing that the set of points in U which maximizes u is both open and closed, meaning that if it is nonempty, it must equal all of U. The weak maximum principle is proved first on strict subsolutions, $-\Delta u < 0$, by the first and second derivative tests. It is then proved on harmonic functions by a perturbations of the form $u_\delta(x) = u(x) + \frac{\delta}{2n}|x|^2$ such that $-\Delta u_\delta = -\Delta u(x) - \delta < 0$ for all $\delta > 0$, and then using uniform convergence of u_δ to a harmonic solution.

Theorem 5. By maximum principles, letting $f \in C(U)$ and $g \in C(\partial U)$, there exists at most one $C^2(U) \cap C(\overline{U})$ solution to

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Can similarly prove that for $u, v \in C^2(U) \cap C(\overline{U})$, and

$$\begin{cases} -\Delta u \leq -\Delta v & \text{in } U \\ u \leq v & \text{on } \partial U, \end{cases}$$

that then $u \leq v$ on $\overline{U}$.

Theorem 6. If u is harmonic in $B(0,r)$, then u satisfies the following **regularity estimate** at $x = 0$,

$$|D^k u(0)| \leq \frac{C(d,k)}{r^k |B(0,r)|} \int_{B(0,r)} |u| dx \leq \frac{C(d,k)}{r^k} \sup_{B(0,r)} |u|.$$

This is proved inductively on derivatives of u by using the mean value principle.
Corrolary: Liouville's Property: If u is harmonic and bounded on all of $\mathbb{R}^n$, then u must be constant.
Corrolary: Let $f \in C_c(\mathbb{R}^n)$. Then $\Phi * f$ is the only bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$, up to additive constants.

Define the **Dirichlet energy**

$$J(u) = \int_U \frac{1}{2}|\nabla u|^2 dx$$

for the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Defining the admissible class

$$\mathcal{A} = \{u \in C^2(U) \cap C(\overline{U}) : u = g \text{ on } \partial U\},$$

we consider the variational problem of finding

$$u = \arg\min_{v \in \mathcal{A}} J(v).$$

Theorem 7. Assuming that there exists a solution $u \in \mathcal{A}$ to either the Dirichlet problem or the variational problem, then u solves both problems.

Can prove that if u solves the variational problem, it must solve the Dirichlet problem by letting $\phi \in C_c^2(U)$ be an arbitrary test function, and using that

$$0 = DJ(u)[\phi] = \left.\frac{d}{dt}\right|_{t=0} J(u + t\phi).$$

To prove the reverse direction, we assume that u is a minimizer and use an energy argument to show that $J(u) \geq J(v)$ for all $v \in \mathcal{A}$.

Theorem 8. There exists at most one solution to the Dirichlet/variational problem.

This is proven by taking the difference between any two solutions and using an energy argument to show that the difference must be equivalently zero.

The **Green's kernel** for Poisson's problem is a function $G : U \times U \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta_y G(x,y) = \delta_x(y) & x, y \in U \\ G(x,y) = 0 & x \in U, y \in \partial U. \end{cases}$$

Additionally, the Poisson's kernel is a function $P : U \times \partial U \to \mathbb{R}$ given by

$$P(x,y) = -\nu(y) \cdot \Delta_y G(x,y), \quad x \in U, y \in \partial U,$$

where $\nu(y)$ is the outward facing normal vector to U at y. Hence, if we define

$$u(x) = \int_U G(x,y) f(y) dy + \int_{\partial U} P(x,y) g(y) dS(y),$$

then u solves the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{in } \partial U. \end{cases}$$

In the whole space, the Green's kernel is given by

$$G(x,y) = \Phi(x-y) - \phi^x(y)$$

where ϕ^x satisfies $-\Delta_y \phi^x = 0$ in U and $\phi^x(y) = \Phi(x-y)$ on ∂U.

In the half space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, we can construct the Green's kernel to satisfy the zero boundary data by

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

where $\tilde{x}$ is the reflection of x across the line $x_d = 0$.

Perron's method for existence of solutions to the homogeneous Dirichlet problem with a sufficiently regular boundary relies on finding the maximal subsolution, proving that it is harmonic, and utilizing regularity of ∂U to show that it satisfies the boundary conditions. It is more complicated than later discussed proofs for existence in Sobolev spaces.

2 Heat Equation

We first consider the homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

We use a similarity solution approach that is rotation invariant and preserves the scaling $(t, x) \to (\lambda^2 t, \lambda x)$. Through choice of proper constants to integrate to one, we find the **fundamental solution/heat kernel**

$$\Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

Theorem 9. Let $g \in C(\mathbb{R}^n)$ be bounded. Then,

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

solves $u_t - \Delta u = 0$ and satisfies $u(x, 0) = g(x)$ in a limiting sense.

Notes on heat kernel:

1. For fixed $t > 0$, $\Phi(x, t) \in C_x^\infty C_t^\infty$ so $u(\cdot, t)$ is C^∞ in x and t as well

2. From the smoothing of the heat kernel, we can see that the backwards-time heat equation is typically ill-posed

3. Due to the infinite support of the heat kernel, the heat equation has infinite speeds of propagation

Now we consider the inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u = f(x, t) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

We use Duhamel's principle which relies on the following fact from ODEs: $y'(t) = Ay$ implies that $y(t) = e^{At} y_0$ where e^{At} is a linear operator (a matrix) acting on the initial data. Now, if we add a nonlinear forcing term, we have

$$y'(t) = Ay + f(t) \implies y(t) = e^{At} y_0 + \int_0^t e^{A(t-s) f(s)} ds.$$

For the heat equation, we have already solved $u_t = Lu$ with $Lu = \Delta u$. Adding the force term, we posit the solution

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dyds,$$

which can be proven to solve the PDE and satisfy the initial conditions.

Now, we consider the homogeneous heat equation in a domain, U,

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times (0,\infty) \\ u = g & \text{on } U \times \{t=0\}. \end{cases}$$

We additionally define the parabolic cylinder, $U_t = U \times (0,t]$ and the parabolic boundary $\partial_p U_t = (U \times \{t=0\}) \cup (\partial U \times (0,T])$.

We say that $u \in C_x^2 C_t^1$ is a subsolution to the heat equation if $u_t - \Delta u \leq 0$ in U_t.

Theorem 10. Let $u \in C_x^2 C_t^1(U_T) \cup C(\overline{U_T})$ be a (sub)solution to the heat equation in U_T, then, the **weak maximum principle holds**,

$$\max_{\overline{U_T}} u = \max_{\partial_p U_T} u.$$

This can be argued by finding the first time t^* for which u exceeds the supremum on its boundary, and the spacial maximum x^* where $u(x^*,t^*)$ equals the supremum. Then, at (x^*,t^*), we have that $u_t > 0$, $\Delta u \leq 0$, and $\nabla u = 0$, which contradicts the PDE.

Corrolary. There exists at most one $C_x^2 C_t^1(U_T) \cup C(\overline{U_T})$ solution of the homogeneous Dirichlet IBVP.

We define the energies

$$e(t) = \int_U u^2 dx, \quad E(t) = \int_U |\nabla u|^2 dx.$$

We can prove that both are nonincreasing, which can be used to prove uniqueness of solutions.

Theorem 11. Local Regularity: Define the parabolic cylinder/heat ball

$$C_r(x_0,t_0) = \{(x,t) : t_0 - r^2 < t \leq t_0, |x-x_0| < r\}.$$

If $u \in C_x^2 C_t^1(C_r(0,0))$ solves the heat equation in $C_r(0,0)$, then

$$\sup_{x \in C_{r/2}(0,0)} |\partial_t^k \nabla^l u(x)| \leq \frac{C(n,k,l)}{r^{l+2k}} \sup_{C_r(0,0)} |u|.$$

3 Wave Equation

The homogeneous wave equation is given by

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = h & \text{on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

The wave equation satisfies several symmetries/invariances. Let $L = (\partial_{tt} - c^2 \Delta)$, and suppose that $Lu = 0$ and $Lv = 0$.

1. Translations: $Lu(x - x_0, t - t_0) = 0$
2. Linearity: $L(au + bv) = 0$
3. Rotations: $Lu(Rx, t) = 0$ where R a rotation matrix
4. Time reversal: $Lu(x, -t) = 0$
5. Scaling: $Lu(\lambda x, \lambda t) = 0$ for all $\lambda > 0$

In 1+1 dimensions (1 space, 1 time), we can factor the wave operator

$$\partial_{tt} - c^2 \partial_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x).$$

We use this to define a change of variables, $\xi = x + ct$, $\eta = x - ct$, and can derive **D'Alembert's formula**

$$u(x,t) = \frac{1}{2}(h(x+ct) + h(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy$$

which can be decomposed into a forward and a backward traveling wave.

For the inhomogeneous wave problem, we again use Duhamel's principle and can prove that the solution has the form

$$\begin{aligned} u(x,t) = &\frac{1}{2}(h(x+ct) + h(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy \\ &+ \frac{1}{2c} \int_0^t \int_{x-ct}^{x+ct} f(y,s)dyds. \end{aligned}$$

We now wish to show finite speed of propagation (speed c). Given a point (x_0, t_0) in space-time, define the energy functional

$$E(t) = \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 dx.$$

Can prove, with Cauchy Schwartz and the Cauchy inequality, that $E'(t) \leq 0$, meaning that if u has initial data of 0 on $(x - ct_0, x + ct_0)$, then u will be equivalently zero on the light cone.

The method of **spherical means** allows us to solve the wave equation in arbitrary dimensions. For this, we define the averages over the sphere $B(x,r)$

$$U(x,t;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)dy$$
$$G(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} g(y)dy$$
$$H(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} h(y)dy.$$

They then satisfy the PDE

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G, U_t = H & \text{in } \mathbb{R}_+ \times \{t = 0\}, \end{cases}$$

where we note that the Laplacian in radial coordinates is precisely $U_{rr} + \frac{n-1}{r}U_r$.

In the special case $n = 3$, performing a scaling $\tilde{U} = rU$ (and similarly for H and G), can solve these equations to get **Kirchhoff's formula** for the homogeneous wave equation in $\mathbb{R}^3$ (with $c = 1$),

$$u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} th(y) + g(y) + \nabla g(y) \cdot (y - x)dS(y).$$

From this we see that the value of $u(x,t)$ depends only on the initial data exactly a distance of t away from x.

For the solution in $\mathbb{R}^2$, we look for a solution in $\mathbb{R}^3$ of the form $\overline{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$, and follow similar steps to find **Poisson's formula**

$$u(x,t) = \frac{1}{2|B(x,t)|} \int_{B(x,t)} \frac{tg(y) + t^2h(y) + t\nabla g(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}} dy.$$

Here, instead of the solution depending only on initial data exactly a distance of t from x, it depends on all initial data up to a distance of t from x.

4 Fourier Methods

Let $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and define the **Fourier transform** of f,

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) \exp(-2\pi i \xi \cdot x) dx.$$

We can similarly define the **inverse Fourier transform** of a function $g \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

$$\check{g}(x) = \mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi) \exp(2\pi i \xi \cdot x) d\xi.$$

We call $x \in \mathbb{R}^n$ the physical space and $\xi \in \mathbb{R}^n$ the frequency space.

The **Plancherel identity** tells us that

$$\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2} \implies ||f||_{L^2} = ||\hat{f}||_{L^2}.$$

Properties of the Fourier transform

1. $\mathcal{F}(f(\cdot + y))(\xi) = \exp(2\pi i \xi \cdot y)\hat{f}(\xi)$
2. $\mathcal{F}(f(\cdot) \exp(2\pi i k \cdot))(\xi) = \hat{f}(\xi - k)$
3. $\mathcal{F}(f(\cdot/\lambda))(\xi) = \lambda^n \hat{f}(\xi)$
4. $\mathcal{F}(\exp(2\pi i k \cdot))(\xi) = \delta_k(\xi)$
5. $\mathcal{F}(f + g)(\xi) = \hat{f}(\xi) + \hat{g}(\xi)$
6. $\mathcal{F}(\nabla f)(\xi) = 2\pi i \xi \hat{f}(\xi)$
7. $\mathcal{F}(\Delta f)(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi)$
8. $\mathcal{F}(f * g)(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

Using these facts, we can derive the heat kernel from the Fourier transform of the heat equation. We can solve the wave equation in the frequency domain, however, no nice closed-form inverse exists due to conservation of energy.

Can similarly be used for Schrödinger's equation

$$iu_t + \Delta u = 0$$

with some initial data.

5 Method of Characteristics and Conservation Laws

Generally, we can write a first order PDE in the form of

$$\begin{cases} F(Du, u, x) = 0 & x \in \Omega \\ u = g & x \in \Gamma \subset \Omega. \end{cases}$$

For problems of this form, we may seek solutions by the method of characteristics. We define $X(s)$ to be the characteristics, parametrized by $s \geq 0$, we also define $Z(s) = u(X(s))$ and $P(s) = \nabla u(X(s)) = \nabla Z(s)$. We first note that

$$Z'(s) = \nabla u(X(s)) \cdot X'(s) = P(s) \cdot X'(s).$$

Then, we use the PDE to write that

$$\begin{aligned} 0 &= \frac{d}{ds} F(P(s), Z(s), X(s)) \\ &= P'(s) \cdot D_P F + (P(s) \cdot X'(s)) F_Z + X'(s) \cdot \nabla_X F. \end{aligned}$$

We note that the above nicely factors if we choose our characteristics to satisfy $X'(s) = D_P F$,

$$\begin{aligned} 0 &= P'(s) \cdot D_P F + (P(s) \cdot X'(s)) F_Z + X'(s) \cdot \nabla_X F \\ &= D_P F \cdot (P'(s) + P(s) F_Z + \nabla_X F) \\ \implies P'(s) &= -P(s) F_Z + \nabla_X F. \end{aligned}$$

Putting all of this together, we have the closed, nonlinear, ODE system

$$\begin{cases} X'(s) = D_P F(P(s), Z(s), X(s)) \\ X(0) = x_0 \in \Gamma \\ Z'(s) = X'(s) \cdot P(s) \\ Z(0) = g(x_0) \\ P'(s) = -P(s) F_Z(P(s), Z(s), X(s)) + \nabla_X F(P(s), Z(s), X(s)) \\ P(0) = \nabla g(x_0). \end{cases}$$

If F is quasi-linear, we can write it in the form

$$F(Du, u, x) = b(x, u) \cdot \nabla u + c(x, u),$$

in which case we have the coupled ODE system

$$\begin{cases} X'(s) = D_P F(P(s), Z(s), X(s)) = b(X(s), Z(s)) \\ Z'(s) = X'(s) \cdot P(s) = b(X(s), Z(s)) \cdot P(s) = -c(X(s), Z(s)). \end{cases}$$

We note that this system is independent of P, and the solution for P is unnecessary in solving the PDE $F(Du, u, x) = 0$.

If F is fully linear, we can write it in the form

$$F(Du, u, x) = b(x) \cdot \nabla u + c(x)u,$$

in which case we have the simpler ODE system which can be solved one equation at a time,

$$\begin{cases} X'(s) = b(X(s)) \\ Z'(s) = -c(X(s))Z(s). \end{cases}$$

Using the implicit function theorem, given the condition that $D_P F(P_0, Z_0, X_0) \cdot \nu(X_0) \neq 0$, can prove local existence of solutions and invertibility of characteristics. However, characteristics may eventually cross meaning they cannot be traced back to a unique initial condition.

Consider a scalar conservation law of the form

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Seeking solutions of the form $u(x,t) = u(X(t), t)$, and choosing X such that $X'(t) = F'(u)$, we get that

$$\frac{d}{dt}u(X(t), t) = u_t(X(t), t) + X'(t)u_x(X(t), t) = u_t(X(t), t) + F'(u(X(t), t))u_x(X(t), t) = 0,$$

implying that solutions are constant along characteristics. Additionally,

$$X'(t) = F'(u(X(t), t)) = F'(X(0), 0) \implies X(t) = F'(X(0), 0)t + X(0),$$

telling us that the characteristics are straight lines.

In order to solve the PDE, we require that we can solve for the initial data, $X(0)$, as a function of $X(t)$ and t, say $G(X(t), t)$. If this is possible, then we can explicitly write the solution to the PDE as

$$u(x, t) = u(G(x, t), 0) = g(G(x, t)).$$

Multiplying the conservation law by a test function $\phi \in C_c^1(\mathbb{R} \times (0,\infty))$ and integrating by parts, we say that u is a **weak solution** to the conservation law if

$$\int_0^\infty \int_{\mathbb{R}} \phi_t u + \phi_x F(u) dx dt + \int_{\mathbb{R}} \phi(x,0) g(x) dx = 0.$$

If we suppose that u is a **classical solution** left and right regions, V_l and V_r respectively, separated by a curve $(\gamma(t), t)$, letting ϕ be supported both on V_l and V_r, we can derive the **Rankine Hugoniot** condition on shock speeds

$$\gamma'(t) = \frac{F(u_l) - F(u_r)}{u_l - u_r}$$

where u_l and u_r are the left and right limiting values of u at the discontinuity.

In order to have uniqueness of weak solutions, we also require that solutions satisfy the **entropy condition** at discontinuities given by

$$F'(u_l) > \gamma'(t) > F'(u_r)$$

which tells us that characteristics cannot emanate from a shock. Note that this condition gives us forwards uniqueness of weak solutions but not backwards uniqueness of weak solutions.

6 Hamilton Jacobi Equations

We introduce the Hamilton Jacobi equations

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where H is the Hamiltonian.

In $n = 1$ dimension, if u solves Hamilton Jacobi, then letting $w = u_x$, we have that

$$w_t + H(w) = \partial_x(u_t + H(u_x)) = 0,$$

so u is the antiderivative to the solution of a scalar conservation law.

Choosing $X'(t) = DH(P(t))$, the characteristic equations for the Hamilton Jacobi equations are given by

$$\begin{cases} X'(t) = DH(P(t)) \\ Z'(t) = DH(P(t)) \cdot P(t) - H(P(t)) \\ P'(t) = 0. \end{cases}$$

Now, define a Lagrangian function $L : \mathbb{R}^n \to \mathbb{R}$, and define the optimal control problem

$$u(x,t) = \inf_{\gamma \in \mathcal{A}} \int_0^t L(\gamma'(s))ds + g(\gamma(0))$$

where the admissible class is $\mathcal{A} = \{\gamma \in C^2([0,t], \mathbb{R}^n) : \gamma(t) = x\}$. We can think of this problem as choosing a path γ with fixed endpoint, which minimizes a traveling cost, L, and a terminal cost, g.

Theorem 12. Minimizers of the above optimal control problem, $u(x,t)$, satisfy the **Euler Lagrange equations**,

$$\frac{-d}{dt} D_v L(\gamma'(s)) + D_x L(\gamma'(s)) = 0.$$

Theorem 13. u satisfies the **dynamic programming principle**,

$$u(x,t) = \inf_{\gamma \in \mathcal{A}} \int_{t_0}^t L(\gamma'(s))ds + u(\gamma(t_0), t_0).$$

Theorem 14. The solution to the optimal control problem is given by the **Hopf-Lax formula**

$$u(x,t) = \min_{y \in \mathbb{R}^n} tL\left(\frac{x-y}{t}\right) + g(y).$$

Choosing $\gamma(s) = y + \frac{s}{t}(x-y)$ to be linear, we get that u is bounded above by $tL\left(\frac{x-y}{t}\right) + g(y)$ for any $y \in \mathbb{R}^n$. The reverse direction can be shown directly by Jensen's inequality.

Assume that L is convex and superlinear, $\lim_{|v|\to\infty} L(v)/|v| = \infty$. We then define the **Fenchel transform** or **Legendre transform** of L by

$$L^*(p) = \sup_{v \in \mathbb{R}^n} p \cdot v - L(v).$$

With the given assumptions, this sup is actually a max, so we know there exists some $v^* \in \mathbb{R}^n$ such that

$$L^*(p) = p \cdot v^* - L(v^*).$$

Theorem 15. Suppose $L : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear, and define L^* as before. Then, L^* is a convex map, and L and L^* are convex dual functions, $L = (L^*)^*$. This implies that provided differentiability, the following three statements are equivalent

$$\begin{cases} p \cdot v = L(v) + L^*(p), \\ p = DL(v), \\ v = DL^*(p). \end{cases}$$

The above theorems allow us to solve the original Hamilton Jacobi equation.

Theorem 16. Suppose that $L : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear, and let $H = L^*$ be the Legendre transform of L. Define u by the Hopf-Lax formula,

$$u(x,t) = \min_{y \in \mathbb{R}^n} tL\left(\frac{x-y}{t}\right) + g(y),$$

for some function g, and suppose that u is differentiable at some point (x^*, t^*). Then, u solves the Hamilton Jacobi equation at (x^*, t^*),

$$u_t(x^*, t^*) + H(Du(x^*, t^*)) = 0,$$

with initial data g.

Additionally, can prove that u as given above is Lipschitz continuous and differentiable a.e. in $\mathbb{R}^n \times (0, \infty)$.

7 Sobolev Spaces

Given some function $u \in C^\infty(U)$, we can define the $H^1(U)$ norm of u by

$$||u||_{H^1(U)} = \int_U |u|^2 + |Du|^2 dx.$$

From this, we can abstractly define the Sobolev space $H^1(U)$ by closure of C^∞ functions

$$H^1(U) = \overline{C^\infty(U)}^{H^1(U)}.$$

First, consider $u \in L^1_{\text{loc}}(U)$, where the space $L^1_{\text{loc}}(U)$ specifies that u is L^1 integrable on all compact subsets of U, but not necessarily on the entire space U. Define the linear functional $D^\alpha : L^1_{\text{loc}}(U) \to (C_c^\infty(U))^*$ s.t.

$$\langle D^\alpha u, \phi \rangle = \int_U (-1)^{|\alpha|} u(x) D^\alpha \phi(x) dx,$$

and we call $D^\alpha u$ a **distributional derivative**.

We say that $f \in L^1_{\text{loc}}(U)$ is the α'th **weak derivative** of u if for all $\phi \in C_c^\infty(U)$,

$$\langle f, \phi \rangle = \langle D^\alpha u, \phi \rangle,$$

or in words, we require that $f = D^\alpha u$ is a locally integrable function. Although u may have a distributional derivative, it may not have a weak derivative.

We can now more easily define the **Sobolev spaces**

$$W^{k,p}(U) = \{u \in L^1_{\text{loc}}(U) : D^\alpha u \in L^1_{\text{loc}}(U), \text{ and } ||D^\alpha u||_{L^p(U)} < \infty \text{ for all } |\alpha| < k\},$$

where $H^k(U)$ refers to the Hilbert space $W^{k,2}(U)$.

Similarly, we define the **Sobolev spaces with zero trace**

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{W^{k,p}(U)}$$

where the $W^{k,p}(U)$ norm is essentially given above.

Now, suppose we wish to prove existence of solutions to PDEs in Sobolev spaces. To do this, we need to make sense of agreement with a boundary condition. Since Sobolev functions are only locally integrable, need to be careful about asserting equality of functions on a set of not full measure.

Some useful theorems about Sobolev spaces:

1. $W^{1,p}$ is a Banach space, and $H^1(U)$ is a Hilbert space with inner product $\langle u, v \rangle_{H^1(U)} = \int_U uv + \nabla u \cdot \nabla v dx$

2. C^∞ is dense in $W^{1,p}$ which can be proved by mollification. This allows us to prove facts in C^∞ and use continuation to prove then in $W^{1,p}$

3. If U is a bounded domain and $\partial U \in C^1$, then there exists a bounded linear operator $T : W^{1,p}(U) \to L^P(\partial U)$ such that $Tu = u|_{\partial U}$ for all $u \in W^{1,p} \cap C(\overline{U})$; this is called the **Sobolev trace** and allows us to deal with boundary conditions

4. We have the relation that $W_0^{1,p}(U) = \{u \in W^{1,p}(U) : Tu = 0\}$.

The trace theorem is proved by changing variables so that the boundary is flat.

Theorem 17. Poincaré Inequality: Let U be a bounded domain, and let $u \in W_0^{1,p}(U)$. Then,

$$||u||_{L^p(U)} \leq C_{U,p}||\nabla u||_{L^p(U)}.$$

Back to PDEs, suppose we have a problem of the form $Lu = f$. Multiplying both sides of the PDE by a test function v in the same space as u, and integrating, we get the weak formulation of the PDE which has the form

$$B(u, v) = F(v).$$

Theorem 18. Lax Milgram: Let H be a real Hilbert space, and let F be a **bounded, linear** functional on H. Suppose that $B : H \times H \to \mathbb{R}$ satisfies

1. **Bilinearity:** B is linear in each argument separately

   $$B(au + bv, cw + dy) = acB(u, w) + adB(u, y) + bcB(v, w) + bdB(v, y),$$

2. **boundedness:** there exists $\alpha > 0$ such that

   $$|B(u, v)| \leq \alpha||u||||v||$$

   for all $u, v \in H$,

3. and **coercivity:** there exists $\beta > 0$ such that

   $$B(u, u) \geq \beta||u||^2$$

   for all $u \in H$.

Then, there exists a unique $u \in H$ such that

$$B(u, v) = F(v)$$

for all $v \in H$.

The proof of this theorem relies heavily on the Riesz Representation theorem for which if we show that B is an inner-product, we can associate u with some element f in the dual space H^*, which is the corresponding dual space element for which $F(v) = \langle f, v \rangle$.

With Lax Milgram, we can prove existence of solutions to PDEs in Sobolev spaces. For example, consider the Poisson problem with zero boundary data

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

This PDE can be written in weak form as finding $u \in H^1(U)$ such that for all $v \in H^1(U)$,

$$B(u, v) = \int_U \nabla u \cdot \nabla v dx = \int_U f v dx = F(v).$$

B is bilinear and F is linear by linearity of products and derivatives. Boundedness of B comes from u and v being H^1 functions, and coercivity of B comes from the Poincaré inequality. Additionally, F is bounded by v being $H^1(U)$. Hence, by Lax Milgram, there exists a unique, weak, solution to the PDE problem. Can use mollification to show that this solution is smooth as well.